10. Algebra Theories II

10.1 essentially algebraic theory

Idea: A mathematical structure is *essentially algebraic* if its definition involves partially defined operations satisfying equational laws, where the domain of any given operation is a subset where various other operations happen to be equal.

An actual algebraic theory is one where all operations are total functions.
The most familiar example may be the (strict) notion of category: a small category consists of a set $C0$of objects, a set $C1$ of morphisms, source and target maps $s,t:C1\to C0$ and so on, but composition is only defined for pairs of morphisms where the source of one happens to equal the target of the other.
Essentially algebraic theories can be understood through category theory at least when they are finitary, so that all operations have only finitely many arguments. This gives a generalization of Lawvere theories, which describe finitary algebraic theories.

As the domains of the operations are given by the solutions to equations, they may be understood using the notion of equalizer. So, just as a Lawvere theory is defined using a category with finite products, a finitary essentially algebraic theory is defined using a category with finite limits — or in other words, finite products and also equalizers (from which all other finite limits, including pullbacks, may be derived).

Definition
As alluded to above, the most concise and elegant definition is through category theory. The traditional definition is this:

Definition. An **essentially algebraic theory** or **finite limits theory** is a category that is finitely complete, i.e., has all finite limits. A **model** of an essentially algebraic theory T is a functor

$$F:T\to \mathrm{Set}$$

that is left exact, i.e., preserves all finite limits. A **homomorphism** of models is a natural transformation

$$\alpha:F\to F'$$

between left exact functors $F,F':T\to \mathrm{Set}$. Models of an essentially algebraic theory T and the homomorphisms between them form a category $\mathrm{Mod}(T)=\mathrm{Lex}(T,\mathrm{Set})$.

More generally, for any category with finite limits X, we can define the category of **models of** T **in** X, $\mathrm{Lex}(T,X)$, which has left exact functors $F:T\to X$ as objects and natural transformations between these as morphisms.
However, the finiteness restriction on the limits above is not part of the core concept of an 'essentially algebraic' structure, so one may prefer to call a category with finite limits an **finitary** essentially algebraic theory.

10.2 Lawvere theory:

The notion of *Lawvere theory* is a joint generalization of the notions of group, ring, associative algebra, etc. In his 1963 doctoral dissertation, Bill Lawvere introduced a new categorical method for doing universal algebra, alternative to the usual way of presenting an algebraic concept by means of its logical signature(with generating operations satisfying equational axioms).

The rough idea is to define an algebraic theory as a category with finite products and possessing a "generic algebra" (e.g., a generic group), and then define a model of that theory (e.g., a group) as a product-preserving functor out of that category. This type of category is what is nowadays called a *Lawvere algebraic theory*, or just Lawvere theory.

Definition. A **Lawvere theory** or **finite-product theory** is (equivalently encoded by its syntactic categorywhich is) a category T with finite products in which every object is isomorphic to a finite cartesian power $xn=x\times x\times\cdots\times x$ of a distinguished object x (called the *generic object* for the theory T).

10.3 generalized algebraic theory:

As described in Cartmell's Generalised Algebraic Theories and Contextual Categories, a generalized algebraic theory (GAT) consists of:

1. An algebraic theory of sorts, which may itself be multi-sorted.
2. A collection of operations, each having zero or more arguments and one result. Each n-ary operation is also given with (n+1)-many derived operations of the algebraic theory of sorts, all of the same arity, specifying the sort of each argument and the sort of the result.
3. Equations between pairs of derived operations with the same arity and whose result sorts are provably equal in the algebraic theory of sorts (see example below).

Relationship to Many-Sorted Algebraic Theories
A many-sorted algebraic theory is a GAT whose algebraic theory of sorts has no equations and no operations of arity greater than zero (i.e., has only constants).

Relationship to Essentially Algebraic Theories

Cartmell's paper explains how, for every GAT there is an EAT with the same models and for every EAT there is a GAT with the same models. In this sense they are more or less equivalent in descriptive power.

However (not in Cartmell's paper), there is no notion in the world of EAT's equivalent to a "GAT without sort equations". This is relevant because it yields an interpretation result. Just as the theory of finite-limit categories is an EAT, and one can interpret any EAT in a finite-limit category, so too is the theory of monoidal categories a GAT without sort equations, and one can interpret any[2] GAT without sort equations in a monoidal category.

Relationship to Enriched Categories

When one interprets the EAT of categories in a finite-limit category? *V*, the result is a *V*-internal category. When one interprets the GAT of categories in a monoidal category *V*, the result is an *V*-enriched category.

The theory of internal categories is an essentially algebraic theory (specifically, the theory for which a model is a category with a designated category internal to it). Likewise, the theory of enriched categories is a GAT without sort equations (specifically, the theory for which a model is a category with a category enriched in it).

Globular theory:

A *globular theory* is much like an *algebraic theory* / *Lawvere theory* only that where the former hasobjects labeled by natural numbers, a globular theory has objects labeled by pasting diagrams of globes. The models of "homogeneous" globular theories are precisely the algebras over globular operads.

10.4 universal algebra

Universal algebra is the study of algebraic theories and their models or algebras. Whereas abstract algebra studies groups, rings, modules and so on — that is, models of particular theories — universal algebra is about algebraic or equational theories in general.

Traditionally, the subject studies models of algebraic theories in the category of sets. The category-theoretic approach abstracts the traditional notions, to study models in more general categories. There are two ways of doing this: by using monads and by using Lawvere theories.

As with the category-theoretic understanding of many other branches of mathematics, the advantage of doing things this way is not so much the obtaining of new results as the unification of many previously disparate points of view. Examples might include how a Hopf algebra is the same thing as a model in a category of vector spaces of the theory of groups, or how computational side-effects in the theory of programming languages may be understood in terms of free algebras?.

Category:

- A category consists of a collection of things and binary relationships (or transitions) between them, such that these relationships can be combined and include the "identity" relationship "is the same as."
- A category is a quiver (a directed graph with multiple edges) with a rule saying how to *compose* two edges that fit together to get a new edge. Furthermore, each vertex has an edge starting and ending at that vertex, which acts as an identity for this composition.
- A category is a combinatorial model for a directed space – a "directed homotopy 1-type" in some sense. It has "points", called *objects*, and also directed "paths", or "processes" connecting these points, called *morphisms*. There is a rule for how to compose paths; and for each object there is an identity path that starts and ends there.
- More precisely, a category consists of a collections of objects and a collection of morphisms. Every morphism has a source object and a target object. If f is a morphism with x as its source and y as its target, we write

$$f:x\to y$$

and we say that f is a morphism from x to y. In a category, we can compose a morphism $g:x\to y$and a morphism $f:y\to z$ to get a morphism $f\circ g:x\to z$. Composition is associative and satisfies the left and right unit laws.

The example to keep in mind is the category Set, in which the objects are sets and a morphism $f:x\to y$is a function from the set x to the set y. Here composition is the usual composition of functions.

There are two broad ways to write down the definition of category; in the usual foundations of mathematics, these two definitions are equivalent. It is good to know both, for several reasons:

- Each introduces its own system of notation, both of which are useful in other parts of category theory, so one should know them.
- One definition generalises quite nicely to the notion of internal category, while the other generalises quite nicely to the notion of enriched category; these are both important concepts.
- When examining alternative foundations, sometimes one definition or the other may be more appropriate; in any case, one will want to examine the question of their equivalence.

The two definitions may be distinguished by whether they use a single collection of all morphisms or several collections of morphisms, a family of collections indexed by pairs of objects.

Size issues

We said a category has a 'collection' of objects and 'collection'(s) of morphisms. A category is said to be small if these collections are all sets — as opposed to proper classes, for example. (The alternatives depend on ones foundations for mathematics.)

Similarly, a category is locally small if $C1(x,y)$ is a set for every pair of objects x,y in that category. The most common motivating examples of categories are all locally small but not small (unless one restricts their objects in some way).

The definition, of a category as a family $C1(x,y)$ of collections of morphisms, generalises to the notion of enriched category: we define a category enriched over (some other category) D as above, with the collection of objects still a 'collection' as before, but with objects of D in place of the collections of morphisms and morphisms of D in place of the various functions. In particular, a category enriched overSet is the same thing as a locally small category.

The classic example of a category is Set, the category with sets as objects and functions as morphisms, and the usual composition of functions as composition. Here are some other famous examples, which arise as variations on this theme:

- Vect - vector spaces as objects, linear maps as morphisms.
- Grp - groups as objects, homomorphisms as morphisms.
- Top - topological spaces as objects, continuous functions as morphisms.
- Diff - smooth manifolds as objects, smooth maps as morphisms.
- Ring - rings as objects, ring homomorphisms as morphisms.

Note that in all these cases the morphisms are actually special sorts of functions. these are concrete categories. That need not be the case in general!

These classic examples are the original motivation for the term "category": all of the above categories encapsulate one "kind of mathematical structure". These are often called "concrete" categories (that term also has a technical definition that these examples all satisfy). But just as widespread in applications as these categorization examples of categories are are other categories (often "small" ones) which, roughly, model something like *states* and *processes* of some system.

- **Poset** A poset can be thought of as a category with its elements as objects and one morphism in each hom(x,y) if x is less than or equal to y, but none otherwise.
- **Group** A group is just a category where there's one object and all the morphisms have inverses - we call the morphisms "elements" of the group. This may seem weird, but it's actually a very useful viewpoint. Here's another way to say it: *A group is a groupoid with a single object*.
- **Monoid** More generally, a monoid is a category with a single object. In fact, this is one way to motivate the concept of categories: categories are the many object version of monoids.
- **Groupoid** A groupoid is a category in which all morphisms are isomorphisms.
- **Quiver** A quiver may be identified with the free category on its directed graph. Given a directed graph G with collection of vertices $G0$ and collection of edges $G1$, there is the *free category* $F(G)$on the graph whose collection of objects coincides with the collection of vertices, and whose

collection of morphisms consists of finite sequences of edges in $G1$ that fit together head-to-tail. The composition operation in this free category is the concatenation of sequences of edges.

- **Universal structure** A category bearing a structure making it initial (or 2-initial) in some doctrine. Examples include the permutation category as the free symmetric monoidal category generated by a single object, or the simplex category which is initial among monoidal categories equipped with a monoid.