

7. Algebraic Equations

7.1 Defined

The study of algebraic equations is probably as old as mathematics: the Babylonian mathematicians, as early as 2000 BC could solve some kind of quadratic equations (displayed on Old Babylonian clay tablets).

The algebraic equations over the rationals with only one variable are also called univariate equations. They have a very long history. Ancient mathematicians wanted the solutions in the form of radical expressions. The ancient Egyptians knew how to solve equations of degree 2 in this manner.

In the 9th century Muhammad ibn Musa al-Khwarizmi and other Islamic mathematicians derived the general quadratic formula and recognized the importance of the discriminant. During the Renaissance, Gerolamo Cardano found the solution to equations of degree 3 and Lodovico Ferrari solved equations of degree 4. Finally Niels Henrik Abel proved, in 1824, that equations of degree 5 and equations of higher degree are not always solvable using radicals. Galois theory, named after Évariste Galois, was introduced to give criteria deciding if an equation is solvable using radicals.

In mathematics, an **algebraic equation** or **polynomial equation** is an equation of the form $P=Q$ where P and Q are polynomials with coefficients in some field, often the field of the rational numbers. For most authors, an algebraic equation is *univariate*, which means that it involves only one variable. On the other hand, a polynomial equation may involve several variables, in which case it is called *multivariate* and the term *polynomial equation* is usually preferred to *algebraic equation*.

For example, $x^5 - 3x + 1 = 0$ is an algebraic equation with integer coefficients.

Some but not all polynomial equations with rational coefficients have a solution that is an algebraic expression with a finite number of operations involving just those coefficients (that is, can be solved algebraically). This can be done for all such equations of degree one, two, three, or four; but for degree five or more it can only be done for some equations but not for all.

A large amount of research has been devoted to compute efficiently accurate approximations of the real or complex solutions of an univariate algebraic equation and of the common solutions of several multivariate polynomial equations.

The algebraic equations are the basis of a number of areas of modern mathematics: Algebraic number theory is the study of (univariate) algebraic equations over the rationals. Galois theory has been introduced by Évariste Galois for getting criteria deciding if an algebraic equation may be solved in terms of radicals. In field theory, an algebraic extension is an extension such that every element is a root of an algebraic equation over the base field. Transcendence theory is the study of the real numbers which are not solutions to an algebraic equation over the rationals. A Diophantine equation is a (usually multivariate) polynomial equation with integer coefficients for which one is interested in the integer solutions. Algebraic geometry is the study of the solutions in an algebraically closed field of multivariate polynomial equations.

As for any equation, the solutions of an equation are the values of the variables for which the equation is true. For univariate algebraic equations these are also called **roots**, even if, properly speaking, one should say *the solutions of the algebraic equation $P=0$ are the roots of the polynomial P* . When solving an equation, it is important to specify in which set the solutions are allowed. For example, for an equation over the rationals one may look for solutions in which all the variables are integers. In this case the equation is a diophantine equation. One may also be interested only in the real solutions. However, for univariate algebraic equations, the number of solutions is finite and all solutions, are contained in any algebraically closed field containing the coefficients, for example, the field of complex numbers in case of equations over the rationals. It follows that without precision "root" and "solution" usually mean "solution in an algebraically closed field".

7.2 Polynomial

In mathematics, a **polynomial** is an expression consisting of variables (or indeterminates) and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponents. An example of a polynomial of a single indeterminate (or variable), x , is $x^2 - 4x + 7$, which is a quadratic polynomial.

A polynomial can either be zero or can be written as the sum of a finite number of non-zero terms. Each term consists of the product of a number—called the coefficient of the term—and a finite number of indeterminates, raised to integer powers. The exponent on an indeterminate in a term is called the degree of that indeterminate in that term; the degree of the term is the sum of the degrees of the indeterminates in that term, and the degree of a polynomial is the largest degree of any one term with nonzero coefficient. Since $x = x^1$, the degree of an indeterminate

without a written exponent is one. A term and a polynomial with no indeterminates are called respectively a constant term and a constant polynomial; the degree of a constant term and of a nonzero constant polynomial is 0. The degree of the zero polynomial (which has no term) is not defined.

According to the Oxford English Dictionary, *polynomial* succeeded the term *binomial*, and was made simply by replacing the Latin root *bi-* with the Greek *poly-*, which comes from the Greek word for *many*. The word *polynomial* was first used in the 17th century.

Polynomials appear in a wide variety of areas of mathematics and science. For example, they are used to form polynomial equations, which encode a wide range of problems, from elementary word problems to complicated problems in the sciences; they are used to define polynomial functions, which appear in settings ranging from basic chemistry and physics to economics and social science; they are used in calculus and numerical analysis to approximate other functions. In advanced mathematics, polynomials are used to construct polynomial rings and algebraic varieties, central concepts in algebra and algebraic geometry.

The x occurring in a polynomial is commonly called either a *variable* or an *indeterminate*. When the polynomial is considered for itself, x is a fixed symbol which does not have any value (its value is "indeterminate"). It is thus more correct to call it an "indeterminate". However, when one considers the function defined by the polynomial, then x represents the argument of the function, and is therefore called a "variable". Many authors use these two words indifferently, but this may be sometimes confusing and is not done in this article.

It is a common convention to use upper case letters for the indeterminates and the corresponding lower case letters for the variables (arguments) of the associated function.

It may be confusing that a polynomial P in the indeterminate X may appear in the formulas either as P or as $P(X)$.

Normally, the name of the polynomial is P , not $P(X)$. However, if a denotes a number, a variable, another polynomial, or, more generally any expression, then $P(a)$ denotes, by convention, the result of substituting X by a in P . For example, the polynomial P defines the function

$$x \mapsto P(x)$$

In particular, if $a = X$, then the definition of $P(a)$ implies

$$P = P(X).$$

This equality allows writing "let $P(X)$ be a polynomial" as a shorthand for "let P be a polynomial in the indeterminate X ". On the other hand, when it is not necessary to emphasize the name of the indeterminate, many formulas are much simpler and easier to read if the name(s) of the indeterminate(s) do not appear at each occurrence of the polynomial.

7.3 Algebraic Number

In mathematics, an **algebraic number** is a number that is a root of a non-zero polynomial in one variable with rational coefficients (or equivalently—by clearing denominators—with integer coefficients). Numbers such as π that are not algebraic are said to be transcendental. Almost all real and complex numbers are transcendental. (Here "almost all" has the sense "all but a countable set";

Properties

Algebraic numbers on the complex plane colored by degree. (red=1, green=2, blue=3, yellow=4)

- The set of algebraic numbers is countable (enumerable).^[3]
- Hence, the set of algebraic numbers has Lebesgue measure zero (as a subset of the complex numbers), i.e. "almost all" complex numbers are not algebraic.
- Given an algebraic number, there is a unique monic polynomial (with rational coefficients) of least degree that has the number as a root. This polynomial is called its minimal polynomial. If its minimal polynomial has degree n , then the algebraic number is said to be of *degree* n . An algebraic number of degree 1 is a rational number. A real algebraic number of degree 2 is a quadratic irrational.
- All algebraic numbers are computable and therefore definable and arithmetical.
- The set of real algebraic numbers is linearly ordered, countable, densely ordered, and without first or last element, so is order-isomorphic to the set of rational numbers.

The field of algebraic numbers

Algebraic numbers colored by degree (blue=4, cyan=3, red=2, green=1). The unit circle in black. The sum, difference, product and quotient of two algebraic

numbers is again algebraic (this fact can be demonstrated using the resultant), and the algebraic numbers therefore form a field, sometimes denoted by \mathbf{A} (which may also denote the adele ring) or \mathbf{Q} . Every root of a polynomial equation whose coefficients are *algebraic numbers* is again algebraic. This can be rephrased by saying that the field of algebraic numbers is algebraically closed. In fact, it is the smallest algebraically closed field containing the rationals, and is therefore called the algebraic closure of the rationals.

Related fields

Numbers defined by Radicals

All numbers that can be obtained from the integers using a finite number of integer additions, subtractions, multiplications, divisions, and taking n th roots (where n is a positive integer) are algebraic. The converse, however, is not true: there are algebraic numbers that cannot be obtained in this manner. All of these numbers are solutions to polynomials of degree ≥ 5 . This is a result of Galois theory (see Quintic equations and the Abel–Ruffini theorem). An example of such a number is the unique real root of the polynomial $x^5 - x - 1$ (which is approximately 1.167304).

7.4 Root-Finding Algorithm

A **root-finding algorithm** is a numerical method, or algorithm, for finding a value x such that $f(x) = 0$, for a given function f . Such an x is called a root of the function f .

Finding integer roots or exact algebraic roots are separate problems, whose algorithms have little in common with those discussed here. Finding a root of $f(x) - g(x) = 0$ is the same as solving the equation $f(x) = g(x)$. Here, x is called the *unknown* in the equation. Conversely, any equation can take the canonical form $f(x) = 0$, so equation solving is the same thing as computing (or *finding*) a root of a function.

Numerical root-finding methods use iteration, producing a sequence of numbers that hopefully converge towards a limit (the so-called "fixed point") which is a root. The first values of this series are *initial guesses*. The method computes subsequent values based on the old ones and the function f .

The behaviour of root-finding algorithms is studied in numerical analysis. Algorithms perform best when they take advantage of known characteristics of the given function. Thus an algorithm to find isolated real roots of a low-degree

polynomial in one variable may bear little resemblance to an algorithm for complex roots of a "black-box" function which is not even known to be differentiable. Questions include ability to separate close roots, robustness against failures of continuity and differentiability, reliability despite inevitable numerical errors, and rate of convergence.

Open methods.

Newton's method (and similar derivative-based methods)

Newton's method assumes the function f to have a continuous derivative. Newton's method may not converge if started too far away from a root. However, when it does converge, it is faster than the bisection method, and is usually quadratic. Newton's method is also important because it readily generalizes to higher-dimensional problems. Newton-like methods with higher orders of convergence are the Householder's methods. The first one after Newton's method is Halley's method with cubic order of convergence.

Secant method

Replacing the derivative in Newton's method with a finite difference, we get the secant method. This method does not require the computation (nor the existence) of a derivative, but the price is slower convergence (the order is approximately 1.6). A generalization of the secant method in higher dimensions is Broyden's method.

Interpolation

The secant method also arises if one approximates the unknown function f by linear interpolation. When quadratic interpolation is used instead, one arrives at Muller's method. It converges faster than the secant method. A particular feature of this method is that the iterates x_n may become complex.

Sidi's method allows for interpolation with an arbitrarily high degree polynomial. The higher the degree of the interpolating polynomial, the faster the convergence. Sidi's method allows for convergence with an order arbitrarily close to 2.

Inverse interpolation

The appearance of complex values in interpolation methods can be avoided by interpolating the inverse of f , resulting in the inverse quadratic interpolation

method. Again, convergence is asymptotically faster than the secant method, but inverse quadratic interpolation often behaves poorly when the iterates are not close to the root.

Bracketing methods

Bracketing methods track the end points of an interval containing a root. This allows them to provide absolute error bounds on a root's location when the function is known to be continuous. Bracketing methods require two initial conditions, one on either side of the root.

Bisection method

The simplest root-finding algorithm is the bisection method. It works when f is a continuous function and it requires previous knowledge of two initial guesses, a and b , such that $f(a)$ and $f(b)$ have opposite signs. Although it is reliable, it converges slowly, gaining one bit of accuracy with each iteration.

False position (regula falsi)

The false position method, also called the *regula falsi* method, is like the secant method. However, instead of retaining the last two points, it makes sure to keep one point on either side of the root. The false position method can be faster than the bisection method and will never diverge like the secant method, but fails to converge under some naive implementations. Ridders' method is a variant on the false-position method that also evaluates the function at the midpoint of the interval, giving faster convergence with similar robustness.

Interpolation

Regula falsi is an interpolation method because it approximates the function with a line between two points. Higher degree polynomials can also be used to approximate the function and its root, while bracketing the root. For example, Muller's method can be easily modified so that rather than always keeping that last 3 points, it tracks the last two points to bracket the root and the best current approximation. Such methods combine good average performance with absolute bounds on the worst-case performance.