9. Algebra Theories I

9.1 Defined

An *algebraic* *theory* is a concept in universal algebra that describes a specific type of algebraic gadget, such as groups or rings. An individual group or ring is a *model* of the appropriate theory. Roughly speaking, an algebraic theory consists of a specification of operations and laws that these operations must satisfy.

Traditionally, algebraic theories were described in terms of logical syntax, as first-order theories whose signatures have only function symbols, no relation symbols, and all of whose axioms are universally quantified equations. Such descriptions may be viewed as *presentations* of a theory, analogous to generators and relations presentations of groups. In particular, different logical presentations can lead to equivalent mathematical objects.

In his thesis, Bill Lawvere undertook a more invariant description of (finitary) algebraic theories. Here *al l*the definable operations of an algebraic theory, or rather their equivalence classes modulo the equational axioms imposed by the theory, are packaged together to form the morphisms of a category with finite products, called a Lawvere theory. None of these operations are considered "primitive", so a Lawvere theory doesn't play favorites among operations.

The article Lawvere theory treats the traditional notion of finitary, single-sorted Lawvere theories, with worked examples. The core of the present article is a working out of the precise connection between infinitary (multi-sorted) Lawvere theories and monads.

Basic Intuitions

Intuitively, a Lawvere theory is the "generic category of products equipped with an object *x* of given algebraic type T ". For example, the Lawvere theory of groups is what you get by assuming a category with products and with a group object x inside, and nothing more; x can be considered "the generic group". Every object in the Lawvere theory is a finite power x n of the generic object x . The morphisms x n →x are nothing but the n-ary operations it is possible to define on x .

In other words, if we abstract away from the usual set-theoretic semantics, and consider a model for the theory of groups to be *any* category with finite products together with a specified group object inside, then the Lawvere theory of groups becomes a universal model of the theory, and carries all the information of the theory but independent of a particular presentation. In this way, theories and models of a theory are placed on an equal footing. A model of a Lawvere theory T in a category with products C is nothing but (i.e., is equivalent to) a product-preserving functor $T \to C$; where the generic object x is sent to is the given model of T in C . If T is the Lawvere theory of groups, then a product-preserving functor $T \to Set$ is tantamount to an ordinary group.

The actual categorical construction of a Lawvere theory is described very easily and elegantly: it is the category opposite to the category of (finitely generated) free algebras of the theory. The free algebra on one generator becomes the generic object.

If theories and models are placed on an equal footing, then what feature sets "theories" *per se* apart? In some very abstract sense, any category with products C could be considered a theory, where the C -models in D are product-preserving functors $C \to D$. Sometimes this is a useful point of view, but it is far removed from traditional syntactic considerations. To give a more "honest" answer, we remember that an ordinary (finitary, single-sorted) algebraic theory a la Lawvere is generated from a single object x , and that every other object should be (at least up to isomorphism) a finite power x^n . The exponent n serves to keep track of arities of operations.

The generic "category of arities" n is, in the finitary case, the category opposite to the category of finite sets (opposite because the n appears contravariantly in powers x^n). This is also the Lawvere "theory of equality", or if you prefer the theory generated by an empty signature. The answer to the question "what sets theories apart" is that a Lawvere theory T should come equipped with a product-preserving functor

$$x^{-} : FinSet^{op} \to T$$

that is essentially surjective (each object of T is isomorphic to x^n for some arity n). As we see below, this definition is a cornerstone to a very elegant theory of algebraic theories.

9.2 Extensions

Infinitary operations

Lawvere's program can be extended to cover many theories with infinitary operations as well. In the best-behaved case, one has algebraic theories involving only operations of arity bounded by some cardinal number — or, more precisely, belonging to some arity class — and these can be understood category-theoretically with a suitable generalization of Lawvere theories. In this bounded case, the Lawvere theory can be described by a small category, and the category of models will be very well behaved, in particular it is a locally presentable category. In such cases there is a satisfying duality between syntax and semantics along the lines of Gabriel-Ulmer duality.

Lawvere's program can to some degree be extended further: one can work with Lawvere theories which are locally small (not just small) categories. Here, the theory might not be bounded, but at least there is only a small set of operations of each arity. Examples of such large theories include

- The theory of algebras with arbitrary sums (one model of which is $[0,\infty]$),
- The theory of sup-lattices, in which there is one operation of each arity, and
- The theory of compact Hausdorff spaces, where the operations are parametrized by ultrafilters.

These examples go outside the bounded (small theory) case. Locally small theories in this sense are co-extensive with the notion of monad (on Set): there is a free-forgetful adjunction between Set and the category of models, and algebras of the theory are equivalent to algebras of the monad.

In the worst case, there are algebraic theories where the number of definable operations explodes, so that there may be a proper class of operations of some fixed arity. In these case there are no free algebras, and Lawvere's reformulation no longer applies. An example is the theory of complete Boolean algebras. (Note: category theorists who define a category $U:A\to\text{Set}$ over sets to be algebraic if it ismonadic would therefore not consider the variety of algebras in such cases to be "algebraic").

9.3 Metaphor

Ring theory is a branch of mathematics with a well-developed terminology. A ring *A* determines and is determined by an algebraic theory, whose models are left *A*-modules and whose n-ary operations have the form

$$(x1,\ldots,xn)\to a1x1+\ldots+anxn$$

for some n-tuple $(a1,\ldots,an)$ of elements of *A*. We may call such an algebraic theory **annular**. The pun*model/module* is due to Jon Beck. The notion that an algebraic theory is a generalized ring is often a fertile one, that automatically provides a slew of suggestive terminology and interesting problems. Many fundamental ideas of ring/module-theory are simply the restriction to annular algebraic theories of ideas that apply more widely to algebraic theories and their models. Let us denote the category of models and homomorphisms (in Set) of an algebraic theory *A* by *A*Mod. Then compare the following to their counterparts in ring theory:

Tensor product theory:

If *A* and *B* are algebraic theories, the algebraic theory *A*⊗*B* is characterized by the fact that its models can be identified with *A*-models in *B*Mod, or equivalently as *B*-models in *A*Mod. There are maps of theories *A*→*A*⊗*B* and *B*→*A*⊗*B* which are universal for maps of theories *A*→*C* and *B*→*C* whose images commute, for any theory *C*.

Matrix theory:

Let *A* be a Lawvere theory with generic object *T*. The full subcategory of *A* generated by the cartesian powers of *Tn* is also a Lawvere theory, that we denote by *Mn*(*A*). In the case of an annular theory (the theory of modules over a ring that we also call *A*), this is the construction of *n*×*n* matrices over *A*. If we denote by *Mn* the application of this construction to the initial theory (the theory of sets), then we may identify *Mn*(*A*) with the tensor product theory *Mn*⊗*A*.

It is an amusing exercise to present *Mn* in terms of generating operations and relations between them.

Bimodel:

Let *A* and *B* be algebraic theories. The category [*A*,*B*] of (*A*,*B*)-bimodels and their homomorphisms is the category of *A*-models and homomorphisms in *B*Modop. An alternative description is that is a co-*A*-model in *B*Mod. Each such bimodel *M* determines and is determined by a pair of adjoint functors

Hom*B*(*M*,?): *B*Mod→*A*Mod
M⊗*A*?: *A*Mod→*B*Mod

Composition of such adjoint pairs yields a functor

⊗*B*:[*B*,*C*]×[*A*,*B*]→[*A*,*C*]

The category [*A*,*A*] has a unit object – it would be churlish not to overload our notation yet further by calling it *A*, corresponding to the fact that the free *A*-model on one generator has a canonical co-*A*-structure.

So we have a bicategory; the 0-cells are algebraic theories, the 1-cells are bimodels and the 2-cells are homomorphisms of bimodels. Consider a monad in this bicategory: an algebraic theory A, an (A,A)-bimodel M, and homomorphisms $\eta:A\to M$, $\mu:M\otimes AM\to M$ satisfying the usual rules.

A module of this monad is given by an A-model B together with an action $M\otimes AB\to B$ satisfying the usual rules. It should be clear that such modules are models of an algebraic theory, which we shall confusingly denote by M. This theory is an extension of A by unary operations (the elements of the underlying set of the underlying A-model of the underlying (A,A)-bimodel of the monad). The rules for composing them are given by μ. They satisfy distributive laws over the operations of A given by the co-A-structure of M.

We may overload η to refer both to a homomorphism of bimodels and to a map of algebraic theories. The forgetful functor MMod→AMod has for its left adjoint the functor $M\otimes A$?, but it also has a right adjoint Hom$A$(M,?). So in this case the forgetful functor preserves colimits as well as limits. In fact all maps of theories whose associated forgetful functors have right adjoints must arise from such a monad in the bi-category of bi-models.

I would like some snappier terminology at this point. What should we call these monads in the bicategory of bimodels? If we use words like *algebra* or *monad* our rickety overloaded onomastic scaffolding starts to creak ominously. Put on your hard hats. We are in territory where to discriminate too meticulously between different views of the same thing is to invite fuddlement. And yet we have to hold in our heads that isomorphism is not equality, and that too cavalier an approach to identification can sometimes lead to error.

If A were a ring, then I'd call M an ‘A-algebra’. Unfortunately, that term can also be used for an A-model. Also, even in ring theory, that term is usually only used when A is commutative. One might, following ‘bimodule’ (and ‘bimodel’) say ‘bi-algebra’ in that case, but that also has another meaning. So let's give up in that direction.

But it seems OK to me to call it an ‘A-monad’. —Toby

This fits with the fact that M is an extension of A by unary operations, so one should be reminded of monoids, maybe? —Gavin