INTEGRALS

10.1 Indefinite integral: Integration is an important concept in mathematics and, together with its inverse, differentiation, is one of the two main operations in calculus. The principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late 17th century. Through the fundamental theorem of calculus, which they independently developed, integration is connected with differentiation: if f is a continuous real-valued function defined on a closed interval [a, b], then, once an antiderivative F of f is known, the definite integral of f over that interval is given by

Integrals and derivatives became the basic tools of calculus, with numerous applications in science and engineering. The founders of calculus thought of the integral as an infinite sum of rectangles of infinitesimal width. A rigorous mathematical definition of the integral was given by Bernhard Riemann. It is based on a limiting procedure which approximates the area of a curvilinear region by breaking the region into thin vertical slabs. Beginning in the nineteenth century, more sophisticated notions of integrals began to appear, where the type of the function as well as the domain over which the integration is performed has been generalised.

A line integral is defined for functions of two or three variables, and the interval of integration [a, b] is replaced by a certain curve connecting two points on the plane or in the space. In a surface integral, the curve is replaced by a piece of a surface in the three-dimensional space. Integrals of differential forms play a fundamental role in modern differential geometry. These generalizations of integrals first arose from the needs of physics, and they play an important role in the formulation of many physical laws, notably those of electrodynamics. There are many modern concepts of integration, among these, the most common is based on the abstract mathematical theory known as Lebesgue integration, developed by Henri Lebesgue.

History

Pre-calculus integration

The first documented systematic technique capable of determining integrals is the method of exhaustion of the ancient Greek astronomer Eudoxus (*ca.* 370 BC), which sought to find areas and volumes by breaking them up into an infinite number of shapes for which the area or volume was known. This method was further developed and employed by Archimedes in the 3rd century BC and used to calculate areas for parabolas and an approximation to the area of a circle. Similar methods were independently developed in China around the 3rd century AD by Liu Hui, who used it to find the area of the circle. This method was later used in the 5th century by Chinese father-and-son mathematicians Zu Chongzhi and Zu Geng to find the volume of a sphere (Shea 2007; Katz 2004, pp. 125–126).

The next significant advances in integral calculus did not begin to appear until the 16th century. At this time the work of Cavalieri with his *method of indivisibles*, and work by Fermat, began to lay the foundations of modern calculus, with Cavalieri computing the integrals of x^n up to degree $n = 9$ in Cavalieri's quadrature formula. Further steps were made in the early 17th century byBarrow and Torricelli, who provided the first hints of a connection between integration and differentiation. Barrow provided the first proof of the fundamental theorem of calculus. Wallisgeneralized Cavalieri's method, computing integrals of x to a general power, including negative powers and fractional powers.

Newton and Leibniz

The major advance in integration came in the 17th century with the independent discovery of the fundamental theorem of calculus by Newton and Leibniz. The theorem demonstrates a connection between integration and differentiation. This connection, combined with the comparative ease of differentiation, can be exploited to calculate integrals. In particular, the fundamental theorem of calculus allows one to solve a much broader class of problems. Equal in importance is the comprehensive mathematical framework that both Newton and Leibniz developed. Given the name infinitesimal calculus, it allowed for precise analysis of functions within continuous domains. This framework eventually became modern calculus, whose notation for integrals is drawn directly from the work of Leibniz.

Formalizing integrals

While Newton and Leibniz provided a systematic approach to integration, their work lacked a degree of rigour. Bishop Berkeley memorably attacked the vanishing increments used by Newton, calling them "ghosts of departed quantities". Calculus acquired a firmer footing with the development of limits. Integration was first rigorously formalized, using limits, by Riemann. Although all bounded piecewise continuous functions are Riemann integrable on a bounded interval, subsequently more general functions were considered—particularly in the context of Fourier analysis—to which Riemann's definition does not apply, and Lebesgue formulated a different definition of integral, founded in measure theory (a subfield of real analysis). Other definitions of integral, extending

Riemann's and Lebesgue's approaches, were proposed. These approaches based on the real number system are the ones most common today, but alternative approaches exist, such as a definition of integral as the standard part of an infinite Riemann sum, based on the hyperreal number system.

Historical notation

Isaac Newton used a small vertical bar above a variable to indicate integration, or placed the variable inside a box. The vertical bar was easily confused with $\dot{x}$ or x', which Newton used to indicate differentiation, and the box notation was difficult for printers to reproduce, so these notations were not widely adopted.

The modern notation for the indefinite integral was introduced by Gottfried Leibniz in 1675 (Burton 1988, p. 359; Leibniz 1899, p. 154). He adapted the integral symbol, ∫, from the letter ſ (long s), standing for *summa* (written as *ſumma*; Latin for "sum" or "total"). The modern notation for the definite integral, with limits above and below the integral sign, was first used by Joseph Fourier in *Mémoires* of the French Academy around 1819–20, reprinted in his book of 1822 (Cajori 1929, pp. 249–250; Fourier 1822, §231).

Terminology and notation

The simplest case, the integral over *x* of a real-valued function *f*(*x*), is written as

$$\int f(x)\,dx.$$

The integral sign ∫ represents integration. The *dx* indicates that we are integrating over *x*; *x* is called the variable of integration. Inside the ∫...*dx* is the expression to be integrated, called the*integrand*. In correct mathematical typography, the *dx* is separated from the integrand by a space (as shown). Some authors use an upright *d* (that is, d*x* instead of *dx*). In this case the integrand is the function *f*(*x*). Because there is no domain specified, the integral is called an *indefinite integral*.

When integrating over a specified domain, we speak of a *definite integral*. Integrating over a domain *D* is written as $\int_D f(x)\,dx,$ or $\int_a^b f(x)\,dx$ if the domain is an interval [*a*, *b*] of *x*.

The domain *D* or the interval [*a*, *b*] is called the *domain of integration*.

If a function has an integral, it is said to be *integrable*. In general, the integrand may be a function of more than one variable, and the domain of integration may be an area, volume, a higher-dimensional region, or even an abstract space that does not have a geometric structure in any usual sense (such as a sample space in probability theory).

In the modern Arabic mathematical notation, which aims at pre-university levels of education in the Arab world and is written from right to left, a reflected integral symbol ∫ is used (W3C 2006).

The variable of integration *dx* has different interpretations depending on the theory being used. It can be seen as strictly a notation indicating that *x* is a dummy variable of integration; if the integral is seen as a Riemann sum, *dx* is a reflection of the weights or widths *d* of the intervals of *x*; in Lebesgue integration and its extensions, *dx* is a measure; in non-standard analysis, it is an infinitesimal; or it can be seen as an independent mathematical quantity, a differential form. More complicated cases may vary the notation slightly. In Leibniz's notation, *dx* is interpreted as an infinitesimal change in *x*. Although Leibniz's interpretation lacks rigour, his integration notation is the most common one in use today.

Introduction

Integrals appear in many practical situations. If a swimming pool is rectangular with a flat bottom, then from its length, width, and depth we can easily determine the volume of water it can contain (to fill it), the area of its surface (to cover it), and the length of its edge (to rope it). But if it is oval with a rounded bottom, all of these quantities call for integrals. Practical approximations may suffice for such trivial examples, but precision engineering (of any discipline) requires exact and rigorous values for these elements.

To start off, consider the curve $y = f(x)$ between $x = 0$ and $x = 1$ with $f(x) = \sqrt{x}$. We ask: What is the area under the function *f*, in the interval from 0 to 1?and call this (yet unknown) area the **integral** of *f*. The notation for this integral will be

$$\int_0^1 \sqrt{x}\,dx.$$

As a first approximation, look at the unit square given by the sides $x = 0$ to $x = 1$ and $y = f(0) = 0$ and $y = f(1) = 1$. Its area is exactly 1. As it is, the true value of the integral must be somewhat less. Decreasing the width of the approximation rectangles shall give a better result; so cross the interval in five steps, using the approximation points 0, 1/5, 2/5, and so on to 1. Fit a box for each step using the right end height of each curve piece, thus $\sqrt{(1/5)}$, $\sqrt{(2/5)}$, and so on to $\sqrt{1} = 1$.

Lebesgue integral

Riemann–Darboux's integration (top) and Lebesgue integration (bottom)
It is often of interest, both in theory and applications, to be able to pass to the limit under the integral. For instance, a sequence of functions can frequently be constructed that approximate, in a suitable sense, the solution to a problem. Then the integral of the solution function should be the limit of the integrals of the approximations. However, many functions that can be obtained as limits are not Riemann integrable, and so such limit theorems do not hold with the Riemann integral. Therefore it is of great importance to have a definition of the integral that allows a wider class of functions to be integrated (Rudin 1987).

Such an integral is the Lebesgue integral, that exploits the following fact to enlarge the class of integrable functions: if the values of a function are rearranged over the domain, the integral of a function should remain the same. Thus Henri Lebesgue introduced the integral bearing his name, explaining this integral thus in a letter to Paul Montel:

I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.

Source: (Siegmund-Schultze 2008)
As Folland (1984, p. 56) puts it, "To compute the Riemann integral of f, one partitions the domain [a,b] into subintervals", while in the Lebesgue integral, "one is in effect partitioning the range of f". The definition of the Lebesgue integral thus begins with a measure, μ. In the simplest case, the Lebesgue measure μ(A) of an interval A = [a,b] is its width, b − a, so that the Lebesgue integral agrees with the (proper) Riemann integral when both exist. In more complicated cases, the sets being measured can be highly fragmented, with no continuity and no resemblance to intervals.

Methods for computing integrals
Analytical
The most basic technique for computing definite integrals of one real variable is based on the fundamental theorem of calculus. Let f(x) be the function of x to be

integrated over a given interval [a, b]. Then, find an antiderivative of f; that is, a function F such that F' = f on the interval. Provided the integrand and integral have no singularities on the path of integration, by the fundamental theorem of calculus,

The integral is not actually the antiderivative, but the fundamental theorem provides a way to use antiderivatives to evaluate definite integrals.

The most difficult step is usually to find the antiderivative of f. It is rarely possible to glance at a function and write down its antiderivative. More often, it is necessary to use one of the many techniques that have been developed to evaluate integrals. Most of these techniques rewrite one integral as a different one which is hopefully more tractable. Techniques include:

Integration by substitution
Integration by parts
Inverse function integration
Changing the order of integration
Integration by trigonometric substitution
Tangent half-angle substitution
Integration by partial fractions
Integration by reduction formulae
Integration using parametric derivatives
Integration using Euler's formula
Euler substitution
Differentiation under the integral sign
Contour integration

Alternative methods exist to compute more complex integrals. Many nonelementary integrals can be expanded in a Taylor series and integrated term by term. Occasionally, the resulting infinite series can be summed analytically. The method of convolution using Meijer G-functions can also be used, assuming that the integrand can be written as a product of Meijer G-functions. There are also many less common ways of calculating definite integrals; for instance, Parseval's identity can be used to transform an integral over a rectangular region into an infinite sum. Occasionally, an integral can be evaluated by a trick; for an example of this, see Gaussian integral.

Computations of volumes of solids of revolution can usually be done with disk integration or shell integration.

Specific results which have been worked out by various techniques are collected in the list of integrals.

Symbolic

Main article: Symbolic integration

Many problems in mathematics, physics, and engineering involve integration where an explicit formula for the integral is desired. Extensive tables of integrals have been compiled and published over the years for this purpose. With the spread of computers, many professionals, educators, and students have turned to computer algebra systems that are specifically designed to perform difficult or tedious tasks, including integration. Symbolic integration has been one of the motivations for the development of the first such systems, like Macsyma.

A major mathematical difficulty in symbolic integration is that in many cases, a closed formula for the antiderivative of a rather simple-looking function does not exist. For instance, it is known that the antiderivatives of the functions exp(x2), xx and (sin x)/x cannot be expressed in the closed form involving only rational and exponential functions, logarithm, trigonometric and inverse trigonometric functions, and the operations of multiplication and composition; in other words, none of the three given functions is integrable in elementary functions, which are the functions which may be built from rational functions, roots of a polynomial, logarithm, and exponential functions.

The Risch algorithm provides a general criterion to determine whether the antiderivative of an elementary function is elementary, and, if it is, to compute it. Unfortunately, it turns out that functions with closed expressions of antiderivatives are the exception rather than the rule. Consequently, computerized algebra systems have no hope of being able to find an antiderivative for a randomly constructed elementary function. On the positive side, if the 'building blocks' for antiderivatives are fixed in advance, it may be still be possible to decide whether the antiderivative of a given function can be expressed using these blocks and operations of multiplication and composition, and to find the symbolic answer whenever it exists. The Risch algorithm, implemented in Mathematica and other computer algebra systems, does just that for functions and antiderivatives built from rational functions, radicals, logarithm, and exponential functions.

Some special integrands occur often enough to warrant special study. In particular, it may be useful to have, in the set of antiderivatives, the special functions of physics (like the Legendre functions, the hypergeometric function, the Gamma function, the Incomplete Gamma function and so on — see Symbolic integration for more details). Extending the Risch's algorithm to include such functions is possible but challenging and has been an active research subject.

More recently a new approach has emerged, using D-finite function, which are the solutions of linear differential equations with polynomial coefficients. Most of the elementary and special functions are D-finite and the integral of a D-finite function is also a D-finite function. This provide an algorithm to express the antiderivative of a D-finite function as the solution of a differential equation.

This theory allows also to compute a definite integrals of a D-function as the sum of a series given by the first coefficients and an algorithm to compute any coefficient.[1]

Numerical

The integrals encountered in a basic calculus course are deliberately chosen for simplicity; those found in real applications are not always so accommodating. Some integrals cannot be found exactly, some require special functions which themselves are a challenge to compute, and others are so complex that finding the exact answer is too slow. This motivates the study and application of numerical methods for approximating integrals, which today use floating-point arithmetic on digital electronic computers. Many of the ideas arose much earlier, for hand calculations; but the speed of general-purpose computers like the ENIAC created a need for improvements.