MATRICES

7.1 concepts of matrices: In mathematics, a matrix (plural matrices) is a rectangular array of numbers, symbols, or expressions, arranged in rows and columnsThe individual items in a matrix are called its elements or entries. An example of a matrix with 2 rows and 3 columns is

Matrices of the same size can be added or subtracted element by element. The rule for matrix multiplication, however, is that two matrices can be multiplied only when the number of columns in the first equals the number of rows in the second. A major application of matrices is to represent linear transformations, that is, generalizations of linear functions such as f(x) = 4x. For example, the rotation of vectors in three dimensional space is a linear transformation which can be represented by a rotation matrix R. If v is a column vector (a matrix with only one column) describing the position of a point in space, the product Rv is a column vector describing the position of that point after a rotation. The product of two matrices is a matrix that represents the composition of two linear transformations. Another application of matrices is in the solution of a system of linear equations. If the matrix is square, it is possible to deduce some of its properties by computing its determinant. For example, a square matrix has an inverse if and only if its determinant is not zero. Eigenvalues and eigenvectors provide insight into the geometry of linear transformations.

Applications of matrices are found in most scientific fields. In every branch of physics, including classical mechanics, optics, electromagnetism, quantum mechanics, and quantum electrodynamics, they are used to study physical phenomena, such as the motion of rigid bodies. In computer graphics, they are used to project a 3-dimensional image onto a 2-dimensional screen. In probability theory and statistics, stochastic matrices are used to describe sets of probabilities; for instance, they are used within the PageRank algorithm that ranks the pages in a Google search.Matrix calculus generalizes classical analytical notions such as derivatives and exponentials to higher dimensions.

A major branch of numerical analysis is devoted to the development of efficient algorithms for matrix computations, a subject that is centuries old and is today an expanding area of research. Matrix decomposition methods simplify computations, both theoretically and practically. Algorithms that are tailored to particular matrix structures, such as sparse matrices and near-diagonal matrices, expedite computations in finite element method and other computations. Infinite matrices occur in planetary theory and in atomic theory. A simple example of an infinite

matrix is the matrix representing the derivative operator, which acts on the Taylor series of a function.

Definition

A *matrix* is a rectangular array of numbers or other mathematical objects, for which operations such as addition and multiplication are defined.[5] Most commonly, a matrix over a field *F* is a rectangular array of scalars from *F*. Most of this article focuses on *real* and *complex matrices*, i.e., matrices whose elements are real numbers or complex numbers, respectively.

A matrix, in general sense, represents a collection of information stored or arranged in an orderly fashion. The mathematical concept of a matrix refers to a set of numbers, variables or functions ordered in rows and columns. Such a set then can be defined as a distinct entity, the matrix, and it can be manipulated as a whole according to some basic mathematical rules.

Size

The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an m × n matrix or m-by-n matrix, while m and n are called its dimensions. For example, the matrix A above is a 3 × 2 matrix.

Matrices which have a single row are called row vectors, and those which have a single column are called column vectors. A matrix which has the same number of rows and columns is called a square matrix. A matrix with an infinite number of rows or columns (or both) is called an infinite matrix. In some contexts such as computer algebra programs it is useful to consider a matrix with no rows or no columns, called an empty matrix.

A "matrix" is a grid, or table, of numbers. For instance, the following matrix represents the prices at the store "Nuthin' But Bed Stuff."

	King-sized	Queen-sized	Twin
MATTRESS	$649	$579	$500
BOX SPRING	$350	$250	$200
Fitted sheet	$15	$12	$10

Top sheet	$15	$12	$10
Blanket	$20	$20	$15

(The matrix is the numbers, not the words that label them.)

Of course, these prices could be displayed in a simple list: "King-sized MATTRESS," "Queen-sized mattress," and so on. However, this two-dimensional display makes it much easier to compare the prices of MATTRESSES to box springs, or the prices of king-sized items to queen-sized items, for instance.

Each horizontal list of numbers is referred to as a row; each vertical list is a column. Hence, the list of all mattresses is a row; the list of all king-sized prices is a column. (It's easy to remember which is which if you think of Greek columns, which are big posts that hold up buildings and are very tall and...well, you know...vertical.) This particular matrix has 5 rows and 3 columns. It is therefore referred to as a 5×3 (read, "5 by 3") matrix.

If a matrix has the same number of columns as rows, it is referred to as a square matrix.

3.2 Matrix operations: In mathematics, matrix multiplication is a binary operation that takes a pair of matrices, and produces another matrix. Numbers such as the real or complex numbers can be multiplied according to elementary arithmetic. On the other hand, matrices are arrays of numbers, so there is no unique way to define "the" multiplication of matrices. As such, in general the term "matrix multiplication" refers to a number of different ways to multiply matrices. The key features of any matrix multiplication include: the number of rows and columns the original matrices have (called the "size", "order" or "dimension"), and specifying how the entries of the matrices generate the new matrix.

Like vectors, matrices of any size can be multiplied by scalars, which amounts to multiplying every entry of the matrix by the same number. Similar to the entry-wise definition of adding or subtracting matrices, multiplication of two matrices of the same size can be defined by multiplying the corresponding entries, and this is known as the Hadamard product. Another definition is the Kronecker product of two matrices, to obtain a block matrix.

One can form many other definitions. However, the most useful definition can be motivated by linear equations and linear transformations on vectors, which have numerous applications in applied mathematics, physics, and engineering. This

definition is often called the matrix product.[1][2] In words, if A is an n × m matrix and B is a m × p matrix, their matrix product AB is an n × p matrix, in which the m entries across the rows of A are multiplied with the m entries down the columns of B (the precise definition is below).

This definition is not commutative, although it still retains the associative property and is distributive over entrywise addition of matrices. The identity element of the matrix product is the identity matrix (analogous to multiplying numbers by 1), and a square matrix may have an inverse matrix (analogous to the multiplicative inverse of a number). A consequence of the matrix product is determinant multiplicativity. The matrix product is an important operation in linear transformations, matrix groups, and the theory of group representations and irreps. For large matrices and/or products of more than two matrices, this matrix product can be very time consuming to calculate, so more efficient algorithms to compute the matrix product than the mathematical definition have been developed.

This article will use the following notational conventions: matrices are represented by capital letters in bold, e.g. A, vectors in lowercase bold, e.g. a, and entries of vectors and matrices are italic (since they are scalars), e.g. A and a. Index notation is often the clearest way to express definitions, and will be used as standard in the literature. The i, j entry of matrix A is indicated by (A)ij or Aij, whereas a numerical label (not matrix entries) on a collection of matrices is subscripted only, e.g. A1, A2, etc.

7.2 Inverse matrix: In linear algebra, an n-by-n square matrix A is called invertible (also nonsingular or nondegenerate) if there exists an n-by-n square. A square matrix that is not invertible is called singular or degenerate. A square matrix is singular if and only if its determinant is 0. Singular matrices are rare in the sense that a square matrix randomly selected from a continuous uniform distribution on its entries will almost never be singular.

Non-square matrices (m-by-n matrices for which $m \neq n$) do not have an inverse. However, in some cases such a matrix may have a left inverse or right inverse. If A is m-by-n and the rank of A is equal to n, then A has a left inverse: an n-by-m matrix B such that BA = I. If A has rank m, then it has a right inverse: an n-by-m matrix B such that AB = I.

Matrix inversion is the process of finding the matrix B that satisfies the prior equation for a given invertible matrix A.

While the most common case is that of matrices over the real or complex numbers, all these definitions can be given for matrices over any commutative ring. However, in this case the condition for a square matrix to be invertible is that its determinant is invertible in the ring, which in general is a much stricter requirement than being nonzero. The conditions for existence of left-inverse resp. right-inverse are more complicated since a notion of rank does not exist over rings.

Density

Over the field of real numbers, the set of singular n-by-n matrices, considered as a subset of Rn×n, is a null set, i.e., has Lebesgue measure zero. This is true because singular matrices are the roots of the polynomial function in the entries of the matrix given by the determinant. Thus in the language of measure theory, almost all n-by-n matrices are invertible.

Furthermore the n-by-n invertible matrices are a dense open set in the topological space of all n-by-n matrices. Equivalently, the set of singular matrices is closed and nowhere dense in the space of n-by-n matrices.

In practice however, one may encounter non-invertible matrices. And in numerical calculations, matrices which are invertible, but close to a non-invertible matrix, can still be problematic; such matrices are said to be ill-conditioned.

Methods of matrix inversion

Gaussian elimination

Gauss–Jordan elimination is an algorithm that can be used to determine whether a given matrix is invertible and to find the inverse. An alternative is the LU decomposition which generates upper and lower triangular matrices which are easier to invert.

Newton's method

A generalization of Newton's method as used for a multiplicative inverse algorithm may be convenient, if it is convenient to find a suitable starting seed:

Victor Pan and John Reif have done work that includes ways of generating a starting seed.[2] [3] Byte magazine summarised one of their approaches as follows (box with equations 8 and 9 not shown):[4]-

The Pan-Reif breakthrough consists of the discovery of a simple and reliable way of evaluating B0, the initial approximation to A-1, which can safely be used for

starting Newton's iteration or variants of it. Readers interested in a derivation can consult the references. I give here merely one example of the results. Let me denote the "Hermitian transpose" of A by AH. That is, if A(I,J) is the element in the Ith row and Jth column of the A matrix, then A*(J,I) is the element at the corresponding position in the matrix A*. Here the star denotes complex conjugate (i.e., if an element is , where x and y are real numbers, then the complex conjugate of that element is). If, as is the case to which I have limited all my own calculations, the elements of A are all real, then AH is just the transposed matrix AT of A (wherein elements are interchanged or "reflected" with respect to the main diagonal).
We now introduce a number t, defined by equation 8. In words: We consider the magnitudes of the various elements A(I,J) of the given A matrix that is to be inverted. (In the case of a complex element , its magnitude is . In the case of a real element, it is just its unsigned or absolute value.) We add up the magnitudes of the elements in a given row and record the sum. We do the same for the remaining rows and then compare the sums thus obtained. The largest of these row sums - just a number - is designated . We do the same for the column sums and take the product of these two maxima, designating its reciprocal as the real number t. Finally, we define our initial approximate inverse matrix B0 as shown in equation 9.
That is, the number t multiplies every element of the Hermitian transpose of the A matrix. Pan and Reif give alternative forms, but this will do. And that's all there is to it.
Otherwise, the method may be adapted to use the starting seed from a trivial starting case by using a homotopy to "walk" in small steps from that to the matrix needed, "dragging" the inverses with them:

where and for some terminating N, perhaps followed by another few iterations at A to settle the inverse.
Using this simplistically on real valued matrices would lead the homotopy through a degenerate matrix about half the time, so complex valued matrices should be used to bypass that, e.g. by using a starting seed S that has i in the first entry, 1 on the rest of the leading diagonal, and 0 elsewhere. If complex arithmetic is not directly available, it may be emulated at a small cost in computer memory by replacing each complex matrix element a+bi with a 2×2 real valued submatrix of the form (see square root of a matrix).

Newton's method is particularly useful when dealing with families of related matrices that behave enough like the sequence manufactured for the homotopy above: sometimes a good starting point for refining an approximation for the new

inverse can be the already obtained inverse of a previous matrix that nearly matches the current matrix, e.g. the pair of sequences of inverse matrices used in obtaining matrix square roots by Denman-Beavers iteration; this may need more than one pass of the iteration at each new matrix, if they are not close enough together for just one to be enough. Newton's method is also useful for "touch up" corrections to the Gauss–Jordan algorithm which has been contaminated by small errors due to imperfect computer arithmetic.

3.4 The Gauss-Jordan method