Descriptive Statistics II

4.1 Axioms and Theorems: Axiom vs Theorem

An axiom is a statement that is considered to be true, based on logic; however, it cannot be proven or demonstrated because it is simply considered as self-evident. Basically, anything declared to be true and accepted, but does not have any proof or has some practical way of proving it, is an axiom. It is also sometimes referred to as a postulate, or an assumption.

An axiom's basis for its truth is often disregarded. It simply is, and there is no need to deliberate any further. However, lots of axioms are still challenged by various minds, and only time will tell if they are crackpots or geniuses.

Axioms can be categorized as logical or non-logical. Logical axioms are universally accepted and valid statements, while non-logical axioms are usually logical expressions used in building mathematical theories.

It is much easier to distinguish an axiom in mathematics. An axiom is often a statement assumed to be true for the sake of expressing a logical sequence. They are the principal building blocks of proving statements. Axioms serve as the starting point of other mathematical statements. These statements, which are derived from axioms, are called theorems.

A theorem, by definition, is a statement proven based on axioms, other theorems, and some set of logical connectives. Theorems are often proven through rigorous mathematical and logical reasoning, and the process towards the proof will, of course, involve one or more axioms and other statements which are already accepted to be true.

Theorems are often expressed to be derived, and these derivations are considered to be the proof of the expression. The two components of the theorem's proof are called the hypothesis and the conclusion. It should be noted that theorems are more often challenged than axioms, because they are subject to more interpretations, and various derivation methods.

It is not difficult to consider some theorems as axioms, since there are other statements that are intuitively assumed to be true. However, they are more appropriately considered as theorems, due to the fact that they can be derived via principles of deduction.

Summary:

1. An axiom is a statement that is assumed to be true without any proof, while a theory is subject to be proven before it is considered to be true or false.

2. An axiom is often self-evident, while a theory will often need other statements, such as other theories and axioms, to become valid.

3. Theorems are naturally challenged more than axioms.

4. Basically, theorems are derived from axioms and a set of logical connectives.

5. Axioms are the basic building blocks of logical or mathematical statements, as they serve as the starting points of theorems.

6. Axioms can be categorized as logical or non-logical.

7. The two components of the theorem's proof are called the hypothesis and the conclusion.

An axiom, or postulate, is a premise or starting point of reasoning. As classically conceived, an axiom is a premise so evident as to be accepted as true without controversy. The word comes from the Greek ἀξίωμα (āxīoma) 'that which is thought worthy or fit' or 'that which commends itself as evident.' As used in modern logic, an axiom is simply a premise or starting point for reasoning. Axioms define and delimit the realm of analysis; the relative truth of an axiom is taken for granted within the particular domain of analysis, and serves as a starting point for deducing and inferring other relative truths. No explicit view regarding the absolute truth of axioms is ever taken in the context of modern mathematics, as such a thing is considered to be an irrelevant and impossible contradiction in terms.

In mathematics, the term axiom is used in two related but distinguishable senses: "logical axioms" and "non-logical axioms". Logical axioms are usually statements that are taken to be true within the system of logic they define (e.g., (A and B) implies A), while non-logical axioms (e.g., a + b = b + a) are actually defining properties for the domain of a specific mathematical theory (such as arithmetic). When used in the latter sense, "axiom," "postulate", and "assumption" may be used interchangeably. In general, a non-logical axiom is not a self-evident truth, but rather a formal logical expression used in deduction to build a mathematical theory. As modern mathematics admits multiple, equally "true" systems of logic, precisely the same thing must be said for logical axioms - they both define and are specific to the particular system of logic that is being invoked. To axiomatize a system of knowledge is to show that its claims can be derived from a small, well-understood set of sentences (the axioms). There are typically multiple ways to axiomatize a given mathematical domain.

In both senses, an axiom is any mathematical statement that serves as a starting point from which other statements are logically derived. Within the system they define, axioms (unless redundant) cannot be derived by principles of deduction, nor are they demonstrable by mathematical proofs, simply because they are starting points; there is nothing else from which they logically follow otherwise they would be classified as theorems. However, an axiom in one system may be a theorem in another, and vice versa.

Historical development

Early Greeks

The logico-deductive method whereby conclusions (new knowledge) follow from premises (old knowledge) through the application of sound arguments (syllogisms, rules of inference), was developed by the ancient Greeks, and has become the core principle of modern mathematics.[citation needed] Tautologies excluded, nothing can be deduced if nothing is assumed. Axioms and postulates are the basic assumptions underlying a given body of deductive knowledge. They are accepted without demonstration. All other assertions (theorems, if we are talking about mathematics) must be proven with the aid of these basic assumptions. However, the interpretation of mathematical knowledge has changed from ancient times to the modern, and consequently the terms axiom and postulate hold a slightly different meaning for the present day mathematician, than they did for Aristotle and Euclid.

The ancient Greeks considered geometry as just one of several sciences,[citation needed] and held the theorems of geometry on par with scientific facts. As such, they developed and used the logico-deductive method as a means of avoiding error, and for structuring and communicating knowledge. Aristotle's posterior analytics is a definitive exposition of the classical view.

An "axiom", in classical terminology, referred to a self-evident assumption common to many branches of science. A good example would be the assertion that

When an equal amount is taken from equals, an equal amount results.

At the foundation of the various sciences lay certain additional hypotheses which were accepted without proof. Such a hypothesis was termed a postulate. While the axioms were common to many sciences, the postulates of each particular science were different. Their validity had to be established by means of real-world experience. Indeed, Aristotle warns that the content of a science cannot be successfully communicated, if the learner is in doubt about the truth of the postulates.

The classical approach is well-illustrated by Euclid's Elements, where a list of postulates is given (common-sensical geometric facts drawn from our experience), followed by a list of "common notions" (very basic, self-evident assertions).

Postulates

It is possible to draw a straight line from any point to any other point.
It is possible to extend a line segment continuously in both directions.
It is possible to describe a circle with any center and any radius.
It is true that all right angles are equal to one another.
("Parallel postulate") It is true that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, intersect on that side on which are the angles less than the two right angles.

Common notions

Things which are equal to the same thing are also equal to one another.
If equals are added to equals, the wholes are equal.
If equals are subtracted from equals, the remainders are equal.
Things which coincide with one another are equal to one another.
The whole is greater than the part.

Modern development

A lesson learned by mathematics in the last 150 years is that it is useful to strip the meaning away from the mathematical assertions (axioms, postulates, propositions, theorems) and definitions. One must concede the need for primitive notions, or undefined terms or concepts, in any study. Such abstraction or formalization makes mathematical knowledge more general, capable of multiple different meanings, and therefore useful in multiple contexts. Alessandro Padoa, Mario Pieri, and Giuseppe Peano were pioneers in this movement.

Structuralist mathematics goes further, and develops theories and axioms (e.g. field theory, group theory, topology, vector spaces) without any particular application in mind. The distinction between an "axiom" and a "postulate" disappears. The postulates of Euclid are profitably motivated by saying that they lead to a great wealth of geometric facts. The truth of these complicated facts rests on the acceptance of the basic hypotheses. However, by throwing out Euclid's fifth postulate we get theories that have meaning in wider contexts, hyperbolic geometry for example. We must simply be prepared to use labels like "line" and "parallel" with greater flexibility. The development of hyperbolic geometry taught mathematicians that postulates should be regarded as purely formal statements, and not as facts based on experience.

When mathematicians employ the field axioms, the intentions are even more abstract. The propositions of field theory do not concern any one particular application; the mathematician now works in complete abstraction. There are many examples of fields; field theory gives correct knowledge about them all.

It is not correct to say that the axioms of field theory are "propositions that are regarded as true without proof." Rather, the field axioms are a set of constraints. If any given system of addition and multiplication satisfies these constraints, then one is in a position to instantly know a great deal of extra information about this system.

Modern mathematics formalizes its foundations to such an extent that mathematical theories can be regarded as mathematical objects, and mathematics itself can be regarded as a branch of logic. Frege, Russell, Poincaré, Hilbert, and Gödel are some of the key figures in this development.

In the modern understanding, a set of axioms is any collection of formally stated assertions from which other formally stated assertions follow by the application of certain well-defined rules. In this view, logic becomes just another formal system. A set of axioms should be consistent; it should be impossible to derive a contradiction from the axiom. A set of axioms should also be non-redundant; an assertion that can be deduced from other axioms need not be regarded as an axiom.

It was the early hope of modern logicians that various branches of mathematics, perhaps all of mathematics, could be derived from a consistent collection of basic axioms. An early success of the formalist program was Hilbert's formalization of Euclidean geometry, and the related demonstration of the consistency of those axioms.

In a wider context, there was an attempt to base all of mathematics on Cantor's set theory. Here the emergence of Russell's paradox, and similar antinomies of naïve set theory raised the possibility that any such system could turn out to be inconsistent.

The formalist project suffered a decisive setback, when in 1931 Gödel showed that it is possible, for any sufficiently large set of axioms (Peano's axioms, for example) to construct a statement whose truth is independent of that set of axioms. As a corollary, Gödel proved that the consistency of a theory like Peano arithmetic is an unprovable assertion within the scope of that theory.

It is reasonable to believe in the consistency of Peano arithmetic because it is satisfied by the system of natural numbers, an infinite but intuitively accessible formal system. However, at present, there is no known way of demonstrating the consistency of the modern Zermelo–Fraenkel axioms for set theory. Furthermore, using techniques of forcing (Cohen) one can show that the continuum hypothesis (Cantor) is independent of the Zermelo–Fraenkel axioms. Thus, even this very general set of axioms cannot be regarded as the definitive foundation for mathematics.

In mathematics, a theorem is a statement that has been proven on the basis of previously established statements, such as other theorems—and generally accepted statements, such as axioms. The proof of a mathematical theorem is a logical argument for the theorem statement given in accord with the rules of a deductive system. The proof of a theorem is often interpreted as justification of the truth of the theorem statement. In light of the requirement that theorems be proved, the concept of a theorem is fundamentally deductive, in contrast to the notion of a scientific theory, which is empirical.[2]

Many mathematical theorems are conditional statements. In this case, the proof deduces the conclusion from the hypotheses. In light of the interpretation of proof as justification of truth, the conclusion is often viewed as a necessary consequence of the hypotheses, namely, that the conclusion is true in case the hypotheses are true, without any further assumptions. However, the conditional could be interpreted differently in certain deductive systems, depending on the meanings assigned to the derivation rules and the conditional symbol.

Although they can be written in a completely symbolic form, for example, within the propositional calculus, theorems are often expressed in a natural language such as English. The same is true of proofs, which are often expressed as logically organized and clearly worded informal arguments, intended to convince readers of the truth of the statement of the theorem beyond any doubt, and from which a formal symbolic proof can in principle be constructed. Such arguments are typically easier to check than purely symbolic ones—indeed, many mathematicians would express a preference for a proof that not only demonstrates the validity of a theorem, but also explains in some way why it is obviously true. In some cases, a picture alone may be sufficient to prove a theorem.

Because theorems lie at the core of mathematics, they are also central to its aesthetics. Theorems are often described as being "trivial", or "difficult", or "deep", or even "beautiful". These subjective judgments vary not only from person to person, but also with time: for example, as a proof is simplified or better understood, a theorem that was once difficult may become trivial. On the other hand, a deep theorem may be simply stated, but its proof may involve surprising and subtle connections between disparate areas of mathematics. Fermat's Last Theorem is a particularly well-known example of such a theorem