4. MATRIX

4.1 MATRIX CONCEPT

The history of matrices goes back to ancient times! But the term "matrix" was not applied to the concept until 1850. "Matrix" is the Latin word for womb, and it retains that sense in English. It can also mean more generally any place in which something is formed or produced.

The origins of mathematical matrices lie with the study of systems of simultaneous linear equations. An important Chinese text from between 300 BC and AD 200, *Nine Chapters of the Mathematical Art* (Chiu Chang Suan Shu), gives the first known example of the use of matrix methods to solve simultaneous equations.

In the treatise's seventh chapter, "Too much and not enough," the concept of a determinant first appears, nearly two millennia before its supposed invention by the Japanese mathematician Seki Kowa in 1683 or his German contemporary Gottfried Leibnitz (who is also credited with the invention of differential calculus, separately from but simultaneously with Isaac Newton).

More uses of matrix-like arrangements of numbers appear in chapter eight, "Methods of rectangular arrays," in which a method is given for solving simultaneous equations using a counting board that is mathematically identical to the modern matrix method of solution outlined by Carl Friedrich Gauss (1777-1855), also known as Gaussian elimination.

The term "matrix" for such arrangements was introduced in 1850 by James Joseph Sylvester. In his 1851 paper, Sylvester wrote, "For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding of pth order."

Because Sylvester was interested in the determinant formed from the rectangular array of number and not the array itself (Kline 1990, p. 804), Sylvester used the term "matrix" in its conventional usage to mean "the place from which something else originates" (Katz 1993). Sylvester (1851) subsequently used the term matrix informally, stating "Form the rectangular matrix consisting of n rows and $(n+1)$ columns.... Then all the $n+1$ determinants that can be formed by rejecting any one column at pleasure out of this matrix are identically zero." However, it remained up to Sylvester's collaborator Cayley to use the terminology in its modern form in papers of 1855 and 1858 (Katz 1993).

Sylvester, incidentally, had a (very) brief career at the University of Virginia, which came to an abrupt end after an enraged Sylvester hit a newspaper-reading student with a sword stick and fled the country, believing he had killed the student!

Since their first appearance in ancient China, matrices have remained important mathematical tools. Today, they are used not simply for solving systems of simultaneous linear equations, but also for describing the quantum mechanics of atomic structure, designing computer game graphics, analyzing relationships, and even plotting complicated dance steps!

The elevation of the matrix from mere tool to important mathematical theory owes a lot to the work of female mathematician Olga Taussky Todd (1906-1995), who began by using matrices to analyze vibrations on airplanes during World War II and became the torchbearer for matrix theory.

In mathematics, a **matrix** (plural **matrices**) is a rectangular *array* of numbers, symbols, or expressions, arranged in *rows* and *columns*. The individual items in a matrix are called its *elements* or *entries*. An example of a matrix with 2 rows and 3 columns is

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}.$$

4.2 **Matri Operations**

"**Operations"** is mathematician-ese for "procedures". The four "basic operations" on numbers are addition, subtraction, multiplication, and division.

For matrices, there are three basic row operations; that is, there are three procedures that you can do with the rows of a matrix:

(1) **You can switch rows**: For instance, given the matrix: $\begin{bmatrix} 0 & 0 & 1 & 3 \\ 1 & 2 & 3 & 5 \\ 0 & 1 & -2 & 4 \end{bmatrix}$ ...you can switch the rows around to put the matrix into a nicer row arrangement, like this:

$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ Row-switching is often indicated by drawing arrows, like this:

$$\begin{bmatrix} 0 & 0 & 1 & 3 \\ 1 & 2 & 3 & 5 \\ 0 & 1 & -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

When switching rows around, be careful to copy the <u>entries</u> correctly.

(2) **Row multiplication**: For instance, given the following matrix:

$\begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$ ...you can multiply the first row by –1 to get a positive leading value in the first row:

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

This row multiplication is often indicated by using an arrow with multiplication listed on top of it, like this:

$$\begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{-1R_1} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

The "$-1R_1$" indicates the actual operation. The "–1" says that we multiplied by negative one; the "R_1" says that we were working with the first row. Note that the second and third rows were copied down, unchanged, into the second matrix. The multiplication only applied to the first row, so the entries for the other two rows were just carried along unchanged.

You can multiply by anything you like. For instance, to get a leading 1 in the third row of the previous matrix, you can multiply the third row by a negative one-half:

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Since you weren't doing anything with the first and second rows, those entries were just copied over unchanged into the new matrix.

You can do more than one row multiplication within the same step, so you could have done the two above steps in just one step, like this:

$$\begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \xrightarrow[-\frac{1}{2}R_3]{-1R_1} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

It is a good idea to use some form of notation (such as the arrows and subscripts above) so you can keep track of your work. Matrices are very messy, especially if you're doing them by hand, and notes can make it easier to check your work later. It'll also impress your teacher

(3) **Row addition:** Row addition is similar to the "addition" method for solving systems of linear equations. Suppose you have the following system of equations:

$x + 3y = 1$
$-x + y = 3$ you could start solving this system by adding down the columns to get $4y = 4$:

$$\begin{array}{r} x + 3y = 1 \\ \underline{-x + \ \ y = 3} \\ 4y = 4 \end{array}$$

You can do something similar with matrices. For instance, given the following matrix:

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ -1 & 1 & 5 & -4 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$ R

...you can "reduce" (get more leading zeroes in) the second row by adding the first row to it (the general goal with matrices at this stage being to get a "1" — or "0's" and then a "1" — at the beginning of each matrix row). When you were reducing the two-equation linear system by adding, you drew an "equals" bar across the bottom and added down. When you are using addition on a matrix, you'll need to grab some scratch paper, because you don't want to try to do the work inside the matrix. So add the two rows on your scratch paper:

4.3 **Inverse Matrix**

In linear algebra, an *n*-by-*n* square matrix **A** is called **invertible** (also **nonsingular** or **non-degenerate**) if there exists an *n*-by-*n* square matrix **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

where $\mathbf{I}_n$ denotes the *n*-by-*n* identity matrix and the multiplication used is ordinary matrix multiplication. If this is the case, then the matrix **B** is uniquely determined by **A** and is called the ***inverse*** of **A**, denoted by $\mathbf{A}^{-1}$.

A square matrix that is not invertible is called **singular** or **degenerate**. A square matrix is singular if and only if its determinant is 0. Singular matrices are rare in the sense that a square matrix randomly selected from a continuous uniform distribution on its entries will almost never be singular.

Non-square matrices, (*m*-by-*n* matrices for which $m \neq n$) do not have an inverse. However, in some cases such a matrix may have a left inverse or right inverse. If **A** is *m*-by-*n* and the rank of **A** is equal to *n*, then **A** has a left inverse: an *n*-by-*m* matrix **B** such that **BA** = **I**. If **A** has rank *m*, then it has a right inverse: an *n*-by-*m* matrix **B** such that **AB** = **I**.

Matrix inversion is the process of finding the matrix **B** that satisfies the prior equation for a given invertible matrix **A**.

While the most common case is that of matrices over the real or complex numbers, all these definitions can be given for matrices over any commutative ring; However, in this case the condition for a square matrix to be invertible is that its determinant is invertible in the ring, which in general is a much stricter requirement than being nonzero. The conditions for existence of left-inverse resp. right-inverse are more complicated since a notion of rank does not exist over rings.

4.4 **Gauss-Jordan Method**

Solving three-variable, three-equation linear systems is more difficult, at least initially, than solving the two-variable systems, because the computations involved are more messy. You will need to be very neat in your working, and you should plan to use lots of scratch paper. The method for solving these systems is an extension of the two-variable solving-by-addition method, so make sure you know this method well and can use it consistently correctly.

Though the method of solution is based on addition/elimination, trying to do actual addition tends to get very messy, so there is a systematized method for solving the three-or-more-variables systems. This method is called "Gaussian elimination" (with the equations ending up in what is called "row-echelon form").

Let's start simple, and work our way up to messier examples:

- **Solve the following system of equations.**

$$5x + 4y - z = 0$$
$$10y - 3z = 11$$
$$z = 3$$

It's fairly easy to see how to proceed in this case. I'll just back-substitute the z-value from the third equation into the second equation, solve the result for y, and then plug z and y into the first equation and solve the result for x.

$$10y - 3(3) = 11$$
$$10y - 9 = 11$$
$$10y = 20$$
$$y = 2$$

$$5x + 4(2) - (3) = 0$$
$$5x + 8 - 3 = 0$$
$$5x + 5 = 0$$
$$5x = -5$$
$$x = -1$$

Then the solution is $\mathbf{(x, y, z) = (-1, 2, 3)}$.

The reason this system was easy to solve is that the system was "triangular"; this refers to the equations having the form of a triangle, because of the lower equations containing only the later variables.

$$\begin{aligned} 5x + 4y - z &= 0 \\ 10y - 3z &= 11 \\ z &= 3 \end{aligned}$$

The point is that, in this format, the system is simple to solve. And Gaussian elimination is the method we'll use to convert systems to this upper triangular form, using the row operations.