Mathematics

1.1 **definition of mathematics:** Mathematics is the study of topics such as quantity (numbers), structure, space and change. There is a range of views among mathematicians and philosophers as to the exact scope and definition of mathematics.

Mathematicians seek out patterns and use them to formulate new conjectures. Mathematicians resolve the truth or falsity of conjectures by mathematical proof. When mathematical structures are good models of real phenomena, then mathematical reasoning can provide insight or predictions about nature. Through the use of abstraction and logic, mathematics developed from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Practical mathematics has been a human activity for as far back as written records exist. The research required to solve mathematical problems can take years or even centuries of sustained inquiry.

Rigorous arguments first appeared in Greek mathematics, most notably in Euclid's Elements. Since the pioneering work of Giuseppe Peano (1858–1932), David Hilbert (1862–1943), and others on axiomatic systems in the late 19th century, it has become customary to view mathematical research as establishing truth by rigorous deduction from appropriately chosen axioms and definitions. Mathematics developed at a relatively slow pace until the Renaissance, when mathematical innovations interacting with new scientific discoveries led to a rapid increase in the rate of mathematical discovery that has continued to the present day.

Galileo Galilei (1564–1642) said, "The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word. Without these, one is wandering about in a dark labyrinth." Carl Friedrich Gauss (1777–1855) referred to mathematics as "the Queen of the Sciences". Benjamin Peirce (1809–1880) called mathematics "the science that draws necessary conclusions".

David Hilbert said of mathematics: "We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise." Albert Einstein (1879–1955) stated that "as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."French mathematician Claire Voisin states "There is creative drive in mathematics; it's all about movement trying to express itself."

Mathematics is used throughout the world as an essential tool in many fields, including natural science, engineering, medicine, finance and the social sciences. Applied mathematics, the branch of mathematics concerned with application of mathematical knowledge to other fields, inspires and makes use of new mathematical discoveries, which has led to the development of entirely new mathematical disciplines, such as statistics and game theory. Mathematicians also engage in pure mathematics; or mathematics for its own sake, without having any application in mind. There is no clear line separating pure and applied mathematics, and practical applications for what began as pure mathematics are often discovered.

Evolution

The evolution of mathematics might be seen as an ever-increasing series of abstractions, or alternatively an expansion of subject matter. The first abstraction, which is shared by many animals, was probably that of numbers: the realization that a collection of two apples and a collection of two oranges (for example) have something in common, namely quantity of their members.

Evidenced by tallies found on bone, in addition to recognizing how to count physical objects, prehistoric peoples may have also recognized how to count abstract quantities, like time – days, seasons, years.

More complex mathematics did not appear until around 3000 BC, when the Babylonians and Egyptians began using arithmetic, algebra and geometry for taxation and other financial calculations, for building and construction, and for astronomy. The earliest uses of mathematics were in trading, land measurement, painting and weaving patterns and the recording of time.

In Babylonian mathematics elementary arithmetic (addition, subtraction, multiplication and division) first appears in the archaeological record. Numeracy pre-dated writing and numeral systems have been many and diverse, with the first known written numerals created by Egyptians in Middle Kingdom texts such as the Rhind Mathematical Papyrus.

Between 600 and 300 BC the Ancient Greeks began a systematic study of mathematics in its own right with Greek mathematics.

Mathematics has since been greatly extended, and there has been a fruitful interaction between mathematics andscience, to the benefit of both. Mathematical discoveries continue to be made today. According to Mikhail B. Sevryuk, in the January 2006 issue of the *Bulletin of the American Mathematical Society*, "The number of papers and books included in the *Mathematical Reviews* database since 1940 (the first year of operation of MR) is now more than 1.9 million, and more than 75 thousand items are added to the database each year. The overwhelming majority of works in this ocean contain new mathematical theorems and their proofs."

Etymology

The word *mathematics* comes from the Greek μάθημα (*máthēma*), which, in the ancient Greek language, means "that which is learnt", "what one gets to know", hence also "study" and "science", and in modern Greek just "lesson". The word *máthēma* is derived from μανθάνω (*manthano*), while the modern Greek equivalent is μαθαίνω (*mathaino*), both of which mean "to learn". In Greece, the word for "mathematics" came to have the narrower and more technical meaning "mathematical study" even in Classical times. Its adjective is μαθηματικός(*mathēmatikós*), meaning "related to learning" or "studious", which likewise further came to mean "mathematical". In particular, μαθηματικὴ τέχνη (*mathēmatikḗ tékhnē*), Latin: *ars mathematica*, meant "the mathematical art".

In Latin, and in English until around 1700, the term *mathematics* more commonly meant "astrology" (or sometimes "astronomy") rather than "mathematics"; the meaning gradually changed to its present one from about 1500 to 1800. This has resulted in several mistranslations: a particularly notorious one is Saint Augustine's warning that Christians should beware of *mathematic i*meaning astrologers, which is sometimes mistranslated as a condemnation of mathematicians.

The apparent plural form in English, like the French plural form *les mathématiques* (and the less commonly used singular derivative *la mathématique*), goes back to the Latin neuter plural*mathematica* (Cicero), based on the Greek plural τα μαθηματικά (*ta mathēmatiká*), used by Aristotle (384–322 BC), and meaning roughly "all things mathematical"; although it is plausible that English borrowed only the adjective *mathematic(al)* and formed the noun *mathematics* anew, after the pattern of physics and metaphysics, which were inherited from the Greek. In English, the noun *mathematics* takes singular verb forms. It is often shortened to *maths* or, in English-speaking North America, *math*.

Definitions of mathematics

Aristotle defined mathematics as "the science of quantity", and this definition prevailed until the 18th century. Starting in the 19th century, when the study of mathematics increased in rigor and began to address abstract topics such as group theory and projective geometry, which have no clear-cut relation to quantity and measurement, mathematicians and philosophers began to propose a variety of new definitions. Some of these definitions emphasize the deductive character of much of mathematics, some emphasize its abstractness, some emphasize certain topics within mathematics. Today, no consensus on the definition of mathematics prevails, even among professionals. There is not even consensus on whether mathematics is an art or a science. A great many professional mathematicians take no interest in a definition of mathematics, or consider it indefinable. Some just say, "Mathematics is what mathematicians do."

Three leading types of definition of mathematics are called logicist, intuitionist, and formalist, each reflecting a different philosophical school of thought. All have severe problems, none has widespread acceptance, and no reconciliation seems possible.

An early definition of mathematics in terms of logic was Benjamin Peirce's "the science that draws necessary conclusions" (1870). In the *Principia Mathematica*, Bertrand Russell and Alfred North Whitehead advanced the philosophical program known as logicism, and attempted to prove that all mathematical concepts, statements, and principles can be defined and proven entirely in terms of symbolic logic. A logicist definition of mathematics is Russell's "All Mathematics is Symbolic Logic" (1903).

Intuitionist definitions, developing from the philosophy of mathematician L.E.J. Brouwer, identify mathematics with certain mental phenomena. An example of an intuitionist definition is "Mathematics is the mental activity which consists in carrying out constructs one after the other." A peculiarity of intuitionism is that it rejects some mathematical ideas considered valid according to other definitions. In particular, while other philosophies of mathematics allow objects that can be proven to exist even though they cannot be constructed, intuitionism allows only mathematical objects that one can actually construct.

Formalist definitions identify mathematics with its symbols and the rules for operating on them. Haskell Curry defined mathematics simply as "the science of formal systems". A formal system is a set of symbols, or *tokens*, and some *rules* telling how the tokens may be combined into *formulas*. In formal systems, the word *axiom* has a special meaning, different from the ordinary meaning of "a self-evident truth". In formal systems, an axiom is a combination of tokens that is included in a given formal system without needing to be derived using the rules of the system.

Inspiration, pure and applied mathematics, and aesthetics

Mathematics arises from many different kinds of problems. At first these were found in commerce, land measurement, architecture and later astronomy; today, all sciences suggest problems studied by mathematicians, and many problems arise within mathematics itself. For example, the physicist Richard Feynman invented the path integral formulation of quantum mechanics using a combination of mathematical reasoning and physical insight, and today's string theory, a still-developing scientific theory which attempts to unify the four fundamental forces of nature, continues to inspire new mathematics.

Some mathematics is relevant only in the area that inspired it, and is applied to solve further problems in that area. But often mathematics inspired by one area proves useful in many areas, and joins the general stock of mathematical concepts. A distinction is often made between pure mathematics and applied mathematics. However pure mathematics topics often turn out to have applications, e.g. number theory in cryptography. This remarkable fact that even the "purest" mathematics often turns out to have practical applications is what Eugene Wigner has called "the unreasonable effectiveness of mathematics". As in most areas of study, the explosion of knowledge in the scientific age has led to specialization: there are now hundreds of specialized areas in mathematics and the latest Mathematics Subject Classification runs to 46 pages. Several areas of applied mathematics have merged with related traditions outside of mathematics and become disciplines in their own right, including statistics, operations research, and computer science.

For those who are mathematically inclined, there is often a definite aesthetic aspect to much of mathematics. Many mathematicians talk about the elegance of mathematics, its intrinsic aesthetics and inner beauty. Simplicity and generality are valued. There is beauty in a simple and elegant proof, such as Euclid's proof that there are infinitely many prime numbers, and in an elegant numerical method that speeds calculation, such as the fast Fourier transform. G.H. Hardy in A Mathematician's Apology expressed the belief that these aesthetic considerations are, in themselves, sufficient to justify the study of pure mathematics. He identified criteria such as significance, unexpectedness, inevitability, and economy as factors that contribute to a mathematical aesthetic. Mathematicians often strive to find proofs that are particularly elegant, proofs from "The Book" of God according to Paul Erdős. The popularity of recreational mathematics is another sign of the pleasure many find in solving mathematical questions.

Notation, language, and rigor

Most of the mathematical notation in use today was not invented until the 16th century. Before that, mathematics was written out in words, a painstaking process that limited mathematical discovery. Euler (1707–1783) was responsible for many of the notations in use today. Modern notation makes mathematics much easier for the professional, but beginners often find it daunting. It is extremely compressed: a few symbols contain a great deal of information. Like musical notation, modern mathematical notation has a strict syntax (which to a limited extent varies from author to author and from discipline to discipline) and encodes information that would be difficult to write in any other way.

Mathematical language can be difficult to understand for beginners. Words such as or and only have more precise meanings than in everyday speech. Moreover, words such as open and field have been given specialized mathematical meanings. Technical terms such as homeomorphism and integrable have precise meanings in mathematics. Additionally, shorthand phrases such as iff for "if and only if" belong to mathematical jargon. There is a reason for special notation and technical vocabulary: mathematics requires more precision than everyday speech. Mathematicians refer to this precision of language and logic as "rigor".

Mathematical proof is fundamentally a matter of rigor. Mathematicians want their theorems to follow from axioms by means of systematic reasoning. This is to avoid mistaken "theorems", based on fallible intuitions, of which many instances have occurred in the history of the subject. The level of rigor expected in mathematics has varied over time: the Greeks expected detailed arguments, but at the time of Isaac Newton the methods employed were less rigorous. Problems inherent in the definitions used by Newton would lead to a resurgence of careful analysis and formal proof in the 19th century. Misunderstanding the rigor is a cause for some of the common misconceptions of mathematics. Today, mathematicians continue to argue among themselves about computer-assisted proofs. Since large computations are hard to verify, such proofs may not be sufficiently rigorous.

Axioms in traditional thought were "self-evident truths", but that conception is problematic. At a formal level, an axiom is just a string of symbols, which has an intrinsic meaning only in the context of all derivable formulas of an axiomatic system. It was the goal of Hilbert's program to put all of mathematics on a firm axiomatic basis, but according to Gödel's incompleteness theorem every (sufficiently powerful) axiomatic system has undecidable formulas; and so a final axiomatization of mathematics is impossible. Nonetheless mathematics is often imagined to be (as far as its formal content) nothing but set theory in some axiomatization, in the sense that every mathematical statement or proof could be cast into formulas within set theory.

Mathematics as science

Gauss referred to mathematics as "the Queen of the Sciences". In the original Latin Regina Scientiarum, as well as in German Königin der Wissenschaften, the word corresponding to science means a "field of knowledge", and this was the original meaning of "science" in English, also; mathematics is in this sense a field of knowledge. The specialization restricting the meaning of "science" to natural science follows the rise of Baconian science, which contrasted "natural science" to scholasticism, the Aristotelean method of inquiring from first principles. The role of empirical experimentation and observation is negligible in mathematics, compared to natural sciences such as psychology, biology, or physics. Albert Einstein stated that "as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." More recently, Marcus du Sautoy has called mathematics "the Queen of Science ... the main driving force behind scientific discovery".

Many philosophers believe that mathematics is not experimentally falsifiable, and thus not a science according to the definition of Karl Popper. However, in the 1930s Gödel's incompleteness theorems convinced many mathematicians[who?] that mathematics cannot be reduced to logic alone, and Karl Popper concluded that "most mathematical theories are, like those of physics and biology, hypothetico-deductive: pure mathematics therefore turns out to be much closer to the natural sciences whose hypotheses are conjectures, than it seemed even recently." Other thinkers, notably Imre Lakatos, have applied a version of falsificationism to mathematics itself.

An alternative view is that certain scientific fields (such as theoretical physics) are mathematics with axioms that are intended to correspond to reality. The theoretical physicist J.M. Ziman proposed that science is public knowledge, and thus includes mathematics. Mathematics shares much in common with many fields in the physical sciences, notably the exploration of the logical consequences of assumptions. Intuition and experimentation also play a role in the formulation of conjectures in both mathematics and the (other) sciences. Experimental mathematics continues to grow in importance within mathematics, and computation and simulation are playing an increasing role in both the sciences and mathematics.

The opinions of mathematicians on this matter are varied. Many mathematicians[who?] feel that to call their area a science is to downplay the importance of its aesthetic side, and its history in the traditional seven liberal arts; others[who?] feel that to ignore its connection to the sciences is to turn a blind eye to the fact that the interface between mathematics and its applications in science and engineering has driven much development in mathematics. One way this difference of viewpoint plays out is in the philosophical debate as to whether mathematics is created (as in art) or discovered (as in science). It is common to see universities divided into sections that include a division of Science and Mathematics, indicating that the fields are seen as being allied but that they do not coincide. In practice, mathematicians are typically grouped with scientists at the gross level but separated at finer levels. This is one of many issues considered in the philosophy of mathematics.