EQUILIBRIUM OF A RIGID BODY AND ANALYSIS OF ETRUCTURAS
8.1 equilibrium of a rigid body in a plane: Rigid-body dynamics studies the movement of systems of interconnected bodies under the action of external forces. The assumption that the bodies are rigid, which means that they do not deform under the action of applied forces, simplifies the analysis by reducing the parameters that describe the configuration of the system to the translation and rotation of reference frames attached to each body.[1][2]

The dynamics of a rigid body system is defined by its equations of motion, which are derived using either Newtons laws of motion or Lagrangian mechanics. The solution of these equations of motion defines how the configuration of the system of rigid bodies changes as a function of time. The formulation and solution of rigid body dynamics is an important tool in the computer simulation of mechanical systems.

If a rigid system of particles moves such that the trajectory of every particle is parallel to a fixed plane, the system is said to be constrained to planar movement. In this case, Newton's laws for a rigid system of N particles, Pi, i=1,...,N, simplify because there is no movement in the k direction. Determine the resultant force and torque at a reference point R, to obtain

$$\mathbf{F} = \sum_{i=1}^{N} m_i \mathbf{A}_i, \quad \mathbf{T} = \sum_{i=1}^{N} (\mathbf{r}_i - \mathbf{R}) \times (m_i \mathbf{A}_i),$$

where ri denotes the planar trajectory of each particle.

Rigid body in three dimensions

Orientation or attitude descriptions

Several methods to describe orientations of a rigid body in three dimensions have been developed.

Euler angles

Euler angles, one of the possible ways to describe an orientation

The first attempt to represent an orientation was owed to Leonhard Euler. He imagined three reference frames that could rotate one around the other, and realized that by starting with a fixed reference frame and performing three rotations, he could get any other reference frame in the space (using two rotations to fix the vertical axis and other to fix the other two axes). The values of these three rotations are called Euler angles.

Tait–Bryan angles

These are three angles, also known as yaw, pitch and roll, Navigation angles and Cardan angles. Mathematically they constitute a set of six possibilities inside the twelve possible sets of Euler angles, the ordering being the one best used for describing the orientation of a vehicle such as an airplane. In aerospace engineering they are usually referred to as Euler angles.

A rotation represented by an Euler axis and angle.

Orientation vector

Euler also realized that the composition of two rotations is equivalent to a single rotation about a different fixed axis (Euler's rotation theorem). Therefore the composition of the former three angles has to be equal to only one rotation, whose axis was complicated to calculate until matrices were developed.

Based on this fact he introduced a vectorial way to describe any rotation, with a vector on the rotation axis and module equal to the value of the angle. Therefore any orientation can be represented by a rotation vector (also called Euler vector) that leads to it from the reference frame. When used to represent an orientation, the rotation vector is commonly called orientation vector, or attitude vector.

A similar method, called axis-angle representation, describes a rotation or orientation using a unit vector aligned with the rotation axis, and a separate value to indicate the angle

Orientation matrix

With the introduction of matrices the Euler theorems were rewritten. The rotations were described by orthogonal matrices referred to as rotation matrices or direction cosine matrices. When used to represent an orientation, a rotation matrix is commonly called orientation matrix, or attitude matrix.

The above mentioned Euler vector is the eigenvector of a rotation matrix (a rotation matrix has a unique real eigen value). The product of two rotation matrices is the composition of rotations. Therefore, as before, the orientation can be given as the rotation from the initial frame to achieve the frame that we want to describe.

The configuration space of a non-symmetrical object in n-dimensional space is $SO(n) \times \mathbf{R}^n$. Orientation may be visualized by attaching a basis of tangent vectors to an object. The direction in which each vector points determines its orientation.

Orientation quaternion

Another way to describe rotations is using rotation quaternions, also called versors. They are equivalent to rotation matrices and rotation vectors. With respect to rotation vectors, they can be more easily converted to and from matrices. When used to represent orientations, rotation quaternions are typically called orientation quaternions or attitude quaternions.

Newton's second law in three dimensions

To consider rigid body dynamics in three-dimensional space, Newton's second law must be extended to define the relationship between the movement of a rigid body and the system of forces and torques that act on it.

Newton's formulated his second law for a particle as, "The change of motion of an object is proportional to the force impressed and is made in the direction of the straight line in which the force is impressed." Because Newton generally referred to mass times velocity as the "motion" of a particle, the phrase "change of motion" refers to the mass times acceleration of the particle, and so this law is usually written as

$$\mathbf{F} = m\mathbf{a},$$

where **F** is understood to be the only external force acting on the particle, *m* is the mass of the particle, and **a** is its acceleration vector. The extension of Newton's second law to rigid bodies is achieved by considering a rigid system of particles.

D'Alembert's form of the principle of virtual work

The equations of motion for a mechanical system of rigid bodies can be determined using D'Alembert's form of the principle of virtual work. The principle of virtual work is used to study the static equilibrium of a system of rigid bodies, however by introducing acceleration terms in Newton's laws this approach is generalized to define dynamic equilibrium.

Static equilibrium

The static equilibrium of a mechanical system rigid bodies is defined by the condition that the virtual work of the applied forces is zero for any virtual displacement of the system. This is known as the principle of virtual work. This is equivalent to the requirement that the generalized forces for any virtual displacement are zero, that is Qi=0.

Let a mechanical system be constructed from n rigid bodies, Bi, i=1,...,n, and let the resultant of the applied forces on each body be the force-torque pairs, Fi and Ti, i=1,...,n. Notice that these applied forces do not include the reaction forces where the bodies are connected. Finally, assume that the velocity Vi and angular velocities ωi, i=,1...,n, for each rigid body, are defined by a single generalized coordinate q. Such a system of rigid bodies is said to have one degree of freedom.

8.2 Rigid Body

In physics, a rigid body is an idealization of a solid body in which deformation is neglected. In other words, the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it. Even though such an object cannot physically exist due to relativity, objects can normally be assumed to be perfectly rigid if they are not moving near the speed of light.

In classical mechanics a rigid body is usually considered as a continuous mass distribution, while in quantum mechanics a rigid body is usually thought of as a collection of point masses. For instance, in quantum mechanics molecules (consisting of the point masses: electrons and nuclei) are often seen as rigid bodies (see classification of molecules as rigid rotors).

Linear and angular position[edit]

The position of a rigid body is the position of all the particles of which it is composed. To simplify the description of this position, we exploit the property that the body is rigid, namely that all its particles maintain the same distance relative to each other. If the body is rigid, it is sufficient to describe the position of at least three non-collinear particles. This makes it possible to reconstruct the position of all the other particles, provided that their time-invariant position relative to the three selected particles is known. However, typically a different, mathematically more convenient, but equivalent approach is used. The position of the whole body is represented by:

the linear position or position of the body, namely the position of one of the particles of the body, specifically chosen as a reference point (typically coinciding with the center of mass or centroid of the body), together with
the angular position (also known as orientation, or attitude) of the body.

Thus, the position of a rigid body has two components: linear and angular, respectively.[2] The same is true for other kinematic and kinetic quantities describing the motion of a rigid body, such as linear and angular velocity, acceleration, momentum, impulse, and kinetic energy.

The linear position can be represented by a vector with its tail at an arbitrary reference point in space (the origin of a chosen coordinate system) and its tip at an arbitrary point of interest on the rigid body, typically coinciding with its center of mass or centroid. This reference point may define the origin of a coordinate system fixed to the body.

There are several ways to numerically describe the orientation of a rigid body, including a set of three Euler angles, a quaternion, or a direction cosine matrix (also referred to as a rotation matrix). All these methods actually define the orientation of a basis set (or coordinate system) which has a fixed orientation relative to the body (i.e. rotates together with the body), relative to another basis set (or coordinate system), from which the motion of the rigid body is observed. For instance, a basis set with fixed orientation relative to an airplane can be defined as a set of three orthogonal unit vectors b1, b2, b3, such that b1 is parallel to the chord line of the wing and directed forward, b2 is normal to the plane of symmetry and directed rightward, and b3 is given by the cross product $b_3 = b_1 \times b_2$.

In general, when a rigid body moves, both its position and orientation vary with time. In the kinematic sense, these changes are referred to as translation and rotation, respectively. Indeed, the position of a rigid body can be viewed as a hypothetic translation and rotation (roto-translation) of the body starting from a hypothetic reference position (not necessarily coinciding with a position actually taken by the body during its motion).

Linear and angular velocity

Velocity (also called linear velocity) and angular velocity are measured with respect to a frame of reference.

The linear velocity of a rigid body is a vector quantity, equal to the time rate of change of its linear position. Thus, it is the velocity of a reference point fixed to the body. During purely translational motion (motion with no rotation), all points on a rigid body move with the same velocity. However, when motion involves rotation, the instantaneous velocity of any two points on the body will generally not be the same. Two points of a rotating body will have the same instantaneous velocity only if they happen to lie on an axis parallel to the instantaneous axis of rotation.

Angular velocity is a vector quantity that describes the angular speed at which the orientation of the rigid body is changing and the instantaneous axis about which it is rotating (the existence of this instantaneous axis is guaranteed by the Euler's rotation theorem). All points on a rigid body experience the same angular velocity at all times. During purely rotational motion, all points on the body change position except for those lying on the instantaneous axis of rotation. The relationship between orientation and angular velocity is not directly analogous to the relationship between position and velocity. Angular velocity is not the time rate of change of orientation, because there is no such concept as an orientation vector that can be differentiated to obtain the angular velocity.

Kinetics

Any point that is rigidly connected to the body can be used as reference point (origin of coordinate system L) to describe the linear motion of the body (the linear position, velocity and acceleration vectors depend on the choice).

However, depending on the application, a convenient choice may be:

the center of mass of the whole system, which generally has the simplest motion for a body moving freely in space;
a point such that the translational motion is zero or simplified, e.g. on an axle or hinge, at the center of a ball and socket joint, etc.
When the center of mass is used as reference point:

The (linear) momentum is independent of the rotational motion. At any time it is equal to the total mass of the rigid body times the translational velocity.
The angular momentum with respect to the center of mass is the same as without translation: at any time it is equal to the inertia tensor times the angular velocity. When the angular velocity is expressed with respect to a coordinate system coinciding with the principal axes of the body, each component of the angular momentum is a product of a moment of inertia (a principal value of the inertia tensor) times the corresponding component of the angular velocity; the torque is the inertia tensor times the angular acceleration.
Possible motions in the absence of external forces are translation with constant velocity, steady rotation about a fixed principal axis, and also torque-free precession.

The net external force on the rigid body is always equal to the total mass times the translational acceleration (i.e., Newton's second law holds for the translational motion, even when the net external torque is nonzero, and/or the body rotates).
The total kinetic energy is simply the sum of translational and rotational energy.

Geometry

Two rigid bodies are said to be different (not copies) if there is no proper rotation from one to the other. A rigid body is called chiral if its mirror image is different in that sense, i.e., if it has either no symmetry or its symmetry group contains only proper rotations. In the opposite case an object is called achiral: the mirror image is a copy, not a different object. Such an object may have a symmetry plane, but not necessarily: there may also be a plane of reflection with respect to which the image of the object is a rotated version. The latter applies for S2n, of which the case n = 1 is inversion symmetry.

For a (rigid) rectangular transparent sheet, inversion symmetry corresponds to having on one side an image without rotational symmetry and on the other side an image such that what shines through is the image at the top side, upside down. We can distinguish two cases:

the sheet surface with the image is not symmetric - in this case the two sides are different, but the mirror image of the object is the same, after a rotation by 180° about the axis perpendicular to the mirror plane.
the sheet surface with the image has a symmetry axis - in this case the two sides are the same, and the mirror image of the object is also the same, again after a rotation by 180° about the axis perpendicular to the mirror plane.
A sheet with a through and through image is achiral. We can distinguish again two cases:

the sheet surface with the image has no symmetry axis - the two sides are different
the sheet surface with the image has a symmetry axis - the two sides are the same